Matrices & Determinant

As for everything else, so for a mathematical theory, beauty can be perceived but not explained..... Cayley Arthur

Introduction :

Any rectangular arrangement of numbers (real or complex) (or of real valued or complex valued expressions) is called a **matrix**. If a matrix has m rows and n columns then the **order** of matrix is written as m × n and we call it as order m by n

The general m × n matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots & a_{1j} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots & a_{2j} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & a_{i3} & \ldots\ldots & a_{ij} & \ldots\ldots & a_{in} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots\ldots & a_{mj} & \ldots\ldots & a_{mn} \end{bmatrix}$$

where a_{ij} denote the element of i^{th} row & j^{th} column. The above matrix is usually denoted as $\mathbf{[a_{ij}]_{m \times n}}$.

Notes :

(i) The elements a_{11}, a_{22}, a_{33},........ are called as **diagonal elements**. Their sum is called as **trace of A** denoted as tr(A)

(ii) Capital letters of English alphabets are used to denote matrices.

(iii) Order of a matrix : If a matrix has m rows and n columns, then we say that its order is "m by n", written as "m × n".

Example # 1 : Construct a 3 × 2 matrix whose elements are given by $a_{ij} = \frac{1}{2} | i - 3j |$.

Solution : In general a 3 × 2 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

$a_{ij} = \frac{1}{2}\ | i - 3j |$, i = 1, 2, 3 and j = 1, 2

Therefore $a_{11} = \frac{1}{2}\ | 1 - 3 \times 1 | = 1$ $\qquad a_{12} = \frac{1}{2} | 1 - 3 \times 2 | = \frac{5}{2}$

$a_{21} = \frac{1}{2}\ | 2 - 3 \times 1 | = \frac{1}{2}$ $\qquad a_{22} = \frac{1}{2}\ | 2 - 3 \times 2 | = 2$

$a_{31} = \frac{1}{2}\ | 3 - 3 \times 1 | = 0$ $\qquad a_{32} = \frac{1}{2}\ | 3 - 3 \times 2 | = \frac{3}{2}$

Hence the required matrix is given hy $A = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{1}{2} & 2 \\ 0 & \frac{3}{2} \end{bmatrix}$

Types of Matrices :

Row matrix :

A matrix having only one row is called as row matrix (or row vector).General form of row matrix is $A = [a_{11}, a_{12}, a_{13}, ..., a_{1n}]$

This is a matrix of order "$1 \times n$" (or a row matrix of order n)

Column matrix :

A matrix having only one column is called as column matrix (or column vector).

Column matrix is in the form $A = \begin{bmatrix} a_{11} \\ a_{21} \\ ... \\ a_{m1} \end{bmatrix}$

This is a matrix of order "$m \times 1$" (or a column matrix of order m)

Zero matrix :

$A = [a_{ij}]_{m \times n}$ is called a zero matrix, if $a_{ij} = 0 \ \forall$ i & j.

e.g. : (i) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Square matrix :

A matrix in which number of rows & columns are equal is called a square matrix. The general form of a square matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{n1} & a_{n2} & .... & a_{nn} \end{bmatrix}$$ which we denote as $A = [a_{ij}]_n$.

This is a matrix of order "$n \times n$" (or a square matrix of order n)

Diagonal matrix :

A square matrix $[a_{ij}]_n$ is said to be a diagonal matrix if $a_{ij} = 0$ for $i \neq j$. (i.e., all the elements of the square matrix other than diagonal elements are zero)

Note : Diagonal matrix of order n is denoted as Diag $(a_{11}, a_{22}, ...a_{nn})$.

e.g. : (i) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ (ii) $\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c \end{bmatrix}$

Scalar matrix :

Scalar matrix is a diagonal matrix in which all the diagonal elements are same. $A = [a_{ij}]_n$ is a scalar matrix, if (i) $a_{ij} = 0$ for $i \neq j$ and (ii) $a_{ij} = k$ for $i = j$.

e.g. : (i) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ (ii) $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

Unit matrix (identity matrix) :

Unit matrix is a diagonal matrix in which all the diagonal elements are unity. Unit matrix of order 'n' is denoted by I_n (or I).

i.e. $A = [a_{ij}]_n$ is a unit matrix when $a_{ij} = 0$ for $i \neq j$ & $a_{ii} = 1$

eg. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Upper triangular matrix :

$A = [a_{ij}]_{m \times n}$ is said to be upper triangular, if $a_{ij} = 0$ for $i > j$ (i.e., all the elements below the diagonal elements are zero).

e.g. : (i) $\begin{bmatrix} a & b & c & d \\ 0 & x & y & z \\ 0 & 0 & u & v \end{bmatrix}$ (ii) $\begin{bmatrix} a & b & c \\ 0 & x & y \\ 0 & 0 & z \end{bmatrix}$

Lower triangular matrix :

$A = [a_{ij}]_{m \times n}$ is said to be a lower triangular matrix, if $a_{ij} = 0$ for $i < j$. (i.e., all the elements above
the diagonal elements are zero.)

e.g. : (i) $\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ x & y & z \end{bmatrix}$ (ii) $\begin{bmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ x & y & z & 0 \end{bmatrix}$

Comparable matrices :

Two matrices A & B are said to be comparable, if they have the same order (i.e., number of rows of A & B are same and also the number of columns).

e.g. : (i) $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & -1 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ are comparable

e.g. : (ii) $C = \begin{bmatrix} 2 & 3 & 4 \\ 3 & -1 & 2 \end{bmatrix}$ & $D = \begin{bmatrix} 3 & 0 \\ 4 & 1 \\ 2 & 3 \end{bmatrix}$ are not comparable

Equality of matrices :

Two matrices A and B are said to be equal if they are comparable and all the corresponding elements are equal.

Let $A = [a_{ij}]_{m \times n}$ & $B = [b_{ij}]_{p \times q}$

$A = B$ iff (i) $m = p$, $n = q$

(ii) $a_{ij} = b_{ij}$ $\forall$ i & j.

Example # 2 : Let $A = \begin{bmatrix} \sin\theta & 1/\sqrt{2} \\ -1/\sqrt{2} & \cos\theta \\ \cos\theta & \tan\theta \end{bmatrix}$ & $B = \begin{bmatrix} 1/\sqrt{2} & \sin\theta \\ \cos\theta & \cos\theta \\ \cos\theta & -1 \end{bmatrix}$. Find θ so that A = B.

Solution : By definition A & B are equal if they have the same order and all the corresponding elements are equal.

Thus we have $\sin\theta = \frac{1}{\sqrt{2}}$, $\cos\theta = -\frac{1}{\sqrt{2}}$ & $\tan\theta = -1$

$\Rightarrow$ $\theta = (2n + 1)\pi - \frac{\pi}{4}$.

Example # 3 : If $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$, then find the values of a, b, c, x, y and z.

Solution : As the given matrices are equal, therefore, their corresponding elements must be equal. Comparing the corresponding elements, we get

$x + 3 = 0$ $\quad z + 4 = 6$ $\quad 2y - 7 = 3y - 2$

$a - 1 = -3$ $\quad 0 = 2c + 2$ $\quad b - 3 = 2b + 4$

$\Rightarrow$ $a = -2, b = -7, c = -1, x = -3, y = -5, z = 2$

Multiplication of matrix by scalar :

Let λ be a scalar (real or complex number) & $A = [a_{ij}]_{m \times n}$ be a matrix. Thus the product λA is defined as $\lambda A = [b_{ij}]_{m \times n}$ where $b_{ij} = \lambda a_{ij}$ $\forall$ i & j.

e.g. : $A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & -1 & -2 \end{bmatrix}$ & $-3A \equiv (-3)\, A = \begin{bmatrix} -6 & 3 & -9 & -15 \\ 0 & -6 & -3 & 9 \\ 0 & 0 & 3 & 6 \end{bmatrix}$

Note : If A is a scalar matrix, then $A = \lambda I$, where λ is a diagonal entry of A

Addition of matrices :

Let A and B be two matrices of same order (i.e. comparable matrices). Then A + B is defined to be.

$A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}$.

$= [c_{ij}]_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij}$ $\forall$ i & j.

e.g. : $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ -2 & -3 \\ 5 & 7 \end{bmatrix}$, $A + B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 6 & 7 \end{bmatrix}$

Substraction of matrices :

Let A & B be two matrices of same order. Then A – B is defined as A + (– B) where – B is (– 1) B.

Properties of addition & scalar multiplication :

Consider all matrices of order m × n, whose elements are from a set F (F denote Q, R or C).
Let $M_{m \times n}$ (F) denote the set of all such matrices.

Then

(a) $A \in M_{m \times n}(F)$ & $B \in M_{m \times n}(F)$ $\Rightarrow$ $A + B \in M_{m \times n}(F)$

(b) $A + B = B + A$

(c) $(A + B) + C = A + (B + C)$

(d) $O = [o]_{m \times n}$ is the additive identity.

(e) For every $A \in M_{m \times n}(F)$, – A is the additive inverse.

(f) $\lambda (A + B) = \lambda A + \lambda B$

(g) $\lambda A = A\lambda$

(h) $(\lambda_1 + \lambda_2)\, A = \lambda_1 A + \lambda_2 A$

Example # 4 : IF $A = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$, then find the matrix X, such that 2A + 3X = 5B

Solution : We have 2A + 3X = 5B.

$\Rightarrow$ $3X = 5B - 2A$

$\Rightarrow$ $X = \frac{1}{3}(5B - 2A)$

$$\Rightarrow \quad X = \frac{1}{3}\left(5\begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix} - 2\begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}\right) = \frac{1}{3}\left(\begin{bmatrix} 10 & -10 \\ 20 & 10 \\ -25 & 5 \end{bmatrix} + \begin{bmatrix} -16 & 0 \\ -8 & 4 \\ -6 & -12 \end{bmatrix}\right)$$

$$\Rightarrow \quad X = \frac{1}{3}\begin{bmatrix} 10-16 & -10+0 \\ 20-8 & 10+4 \\ -25-6 & 5-12 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -6 & -10 \\ 12 & 14 \\ -31 & -7 \end{bmatrix} = \begin{bmatrix} -2 & \frac{-10}{3} \\ 4 & \frac{14}{3} \\ \frac{-31}{3} & \frac{-7}{3} \end{bmatrix}$$

Multiplication of matrices :

Let A and B be two matrices such that the number of columns of A is same as number of rows of B. i.e., $A = [a_{ij}]_{m \times p}$ & $B = [b_{ij}]_{p \times n}$.

Then $AB = [c_{ij}]_{m \times n}$ where $c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}$, which is the dot product of i^{th} row vector of A and j^{th} column vector of B.

e.g. : $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}$, $AB = \begin{bmatrix} 3 & 4 & 9 & 1 \\ 1 & 3 & 7 & 2 \end{bmatrix}$

Notes :

(1) The product AB is defined iff the number of columns of A is equal to the number of rows of B. A is called as premultiplier & B is called as post multiplier. AB is defined BA is defined.

(2) In general $AB \neq BA$, even when both the products are defined.

(3) A (BC) = (AB) C, whenever it is defined.

Properties of matrix multiplication :

Consider all square matrices of order 'n'. Let M_n (F) denote the set of all square matrices of order n. (where F is Q, R or C). Then

(a) $A, B \in M_n(F) \Rightarrow AB \in M_n(F)$

(b) In general $AB \neq BA$

(c) (AB) C = A(BC)

(d) I_n, the identity matrix of order n, is the multiplicative identity.

$AI_n = A = I_n A \quad \forall A \in M_n(F)$

(e) For every non singular matrix A (i.e., $|A| \neq 0$) of M_n (F) there exist a unique (particular) matrix $B \in M_n(F)$ so that $AB = I_n = BA$. In this case we say that A & B are multiplicative inverse of one another. In notations, we write $B = A^{-1}$ or $A = B^{-1}$.

(f) If λ is a scalar $(\lambda A) B = \lambda(AB) = A(\lambda B)$.

(g) $A(B + C) = AB + AC \quad \forall A, B, C \in M_n(F)$

(h) $(A + B) C = AC + BC \quad \forall A, B, C \in M_n(F)$.

Notes : (1) Let $A = [a_{ij}]_{m \times n}$. Then $AI_n = A$ & $I_m A = A$, where I_n & I_m are identity matrices of order n & m respectively.

(2) For a square matrix A, A^2 denotes AA, A^3 denotes AAA etc.

Example # 5 : If A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, then show that $A^3 - 23A - 40\,I = O$

Solution : We have $A^2 = A.A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$

So $A^3 = AA^2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$

Now $A^3 - 23A - 40I = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - 23\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} + \begin{bmatrix} -23 & -46 & -69 \\ -69 & 46 & -23 \\ -92 & -46 & -23 \end{bmatrix} + \begin{bmatrix} -40 & 0 & 0 \\ 0 & -40 & 0 \\ 0 & 0 & -40 \end{bmatrix}$$

$$= \begin{bmatrix} 63-23-40 & 46-46+0 & 69-69+0 \\ 69-69+0 & -6+46-40 & 23-23+0 \\ 90-92+0 & 46-46+0 & 63-23-40 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Self practice problems :

(1) If $A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, verify that $A(\alpha)\,A(\beta) = A(\alpha + \beta)$.

Hence show that in this case $A(\alpha).\ A(\beta) = A(\beta)\ .\ A(\alpha)$.

(2) Let A = $\begin{bmatrix} 4 & 6 & -1 \\ 3 & 0 & 2 \\ 1 & -2 & 5 \end{bmatrix}$, B = $\begin{bmatrix} 2 & 4 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}$ and C = [3 1 2].

Then which of the products ABC, ACB, BAC, BCA, CAB, CBA are defined. Calculate the product whichever is defined.

Answer (2) Only CAB is defined. CAB = [25 100]

Transpose of a matrix :

Let $A = [a_{ij}]_{m \times n}$. Then the transpose of A is denoted by A'(or A^T) and is defined as

$A' = [b_{ij}]_{n \times m}$ where $b_{ij} = a_{ji}\ \ \forall$ i & j.

i.e. A′ is obtained by rewriting all the rows of A as columns (or by rewriting all the columns of A as rows).

e.g. : A = $\begin{bmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \\ x & y & z & w \end{bmatrix}$, A′ = $\begin{bmatrix} 1 & a & x \\ 2 & b & y \\ 3 & c & z \\ 4 & d & w \end{bmatrix}$

Results : (i) For any matrix $A = [a_{ij}]_{m \times n}$, $(A')' = A$

(ii) Let λ be a scalar & A be a matrix. Then $(\lambda A)' = \lambda A'$

(iii) $(A + B)' = A' + B'$ & $(A - B)' = A' - B'$ for two comparable matrices A and B.

(iv) $(A_1 \pm A_2 \pm ..... \pm A_n)' = A_1' \pm A_2' \pm ..... \pm A_n'$, where A_i are comparable.

(v) Let $A = [a_{ij}]_{m \times p}$ & $B = [b_{ij}]_{p \times n}$, then $(AB)' = B'A'$

(vi) $(A_1 A_2 ....A_n)' = A_n'. A_{n-1}' ....A_2' . A_1'$, provided the product is defined.

Symmetric & skew-symmetric matrix : A square matrix A is said to be symmetric if $A' = A$

i.e. Let $A = [a_{ij}]_n$. A is symmetric iff $a_{ij} = a_{ji}$ $\forall$ i & j. A square matrix A is said to be skew-symmetric if $A' = -A$

i.e. Let $A = [a_{ij}]_n$. A is skew-symmetric iff $a_{ij} = -a_{ji}$ $\forall$ i & j.

e.g. $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix.

$B = \begin{bmatrix} o & x & y \\ -x & o & z \\ -y & -z & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Notes :

(1) In a skew-symmetric matrix all the diagonal elements are zero.
($\because$ $a_{ii} = -a_{ii}$ $\Rightarrow$ $a_{ii} = 0$)

(2) For any square matrix A, $A + A'$ is symmetric & $A - A'$ is skew-symmetric.

(3) Every square matrix can be uniquely expressed as a sum of two square matrices of which one is symmetric and the other is skew-symmetric.
$A = B + C$, where $B = \frac{1}{2}(A + A')$ & $C = \frac{1}{2}(A - A')$.

Example # 6 : If $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$, $B = [1 \ \ 3 \ \ -6]$, verify that $(AB)' = B'A'$.

Solution : We have

$$A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}, B = [1 \ \ 3 \ \ -6]$$

Then $$AB = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix} [1 \ \ 3 \ \ -6] = \begin{bmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{bmatrix}$$

Now $$A' = [-2 \ \ 4 \ \ 5], B' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} [-2 \ \ 4 \ \ 5] = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix} = (AB)'$$

Clearly $(AB)' = B'A'$

Example # 7 : Express the matrix B = $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

Solution : Here $B' = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$

Let $P = \frac{1}{2}(B + B') = \frac{1}{2}\begin{bmatrix} 4 & -3 & -3 \\ -3 & 6 & 2 \\ -3 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & 3 & 1 \\ -\frac{3}{2} & 1 & -3 \end{bmatrix}$

Now $P' = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} = P$

Thus $P = \frac{1}{2}(B + B')$ is a symmetric matrix.

Also, Let $Q = \frac{1}{2}(B - B') = \frac{1}{2}\begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$

Now $Q' = \begin{bmatrix} 0 & \frac{1}{2} & \frac{5}{3} \\ -\frac{1}{2} & 0 & -3 \\ -\frac{5}{2} & 3 & 0 \end{bmatrix} = -Q$

Thus $Q = \frac{1}{2}(B - B')$ is a skew symmetric matrix.

Now $P + Q = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = B$

Thus, B is represented as the sum of a symmetric and a skew symmetric matrix.

Thus, B is represented as the sum of a symmetric and a skew symmetric matrix.

Example # 8 : Show that BAB′ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric (where B is any square matrix whose order is same as that of A).

Solution : Case - I A is symmetric $\Rightarrow$ A′ = A

(BAB′)′ = (B′)′A′B′ = BAB′ $\Rightarrow$ BAB′ is symmetric.

Case - II A is skew-symmetric $\Rightarrow$ A′ = – A

(BAB′)′ = (B′)′A′B′

= B (– A) B′

= – (BAB′)

$\Rightarrow$ BAB′ is skew-symmetric

Self practice problems :

(3) For any square matrix A, show that A′A & AA′ are symmetric matrices.

(4) If A & B are symmetric matrices of same order, then show that AB + BA is symmetric and AB – BA is skew-symmetric.

Submatrix : Let A be a given matrix. The matrix obtained by deleting some rows or columns of A is called as submatrix of A.

eg. $A = \begin{bmatrix} a & b & c & d \\ x & y & z & w \\ p & q & r & s \end{bmatrix}$ Then $\begin{bmatrix} a & c \\ x & z \\ p & r \end{bmatrix}$, $\begin{bmatrix} a & b & d \\ p & q & s \end{bmatrix}$, $\begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}$ are all submatrices of A.

Determinant of a square matrix :

To every square matrix A = $[a_{ij}]$ of order n, we can associate a number (real or complex) called determinant of the square matrix.

Let A = $[a]_{1\times1}$ be a 1×1 matrix. Determinant A is defined as |A| = a.

e.g. A = $[-3]_{1\times1}$ |A| = – 3

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then |A| is defined as ad – bc.

e.g. $A = \begin{bmatrix} 5 & 3 \\ -1 & 4 \end{bmatrix}$, |A| = 23

Minors & Cofactors :

Let Δ be a determinant. Then minor of element a_{ij}, denoted by M_{ij}, is defined as the determinant of the submatrix obtained by deleting i^{th} row & j^{th} column of Δ. Cofactor of element a_{ij}, denoted by C_{ij}, is defined as $C_{ij} = (-1)^{i+j} M_{ij}$.

e.g. 1 $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$M_{11} = d = C_{11}$

$M_{12} = c,\ C_{12} = -c$

$M_{21} = b,\ C_{21} = -b$

$M_{22} = a = C_{22}$

e.g. 2 $\Delta = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

$M_{11} = \begin{vmatrix} q & r \\ y & z \end{vmatrix} = qz - yr = C_{11}.$

$M_{23} = \begin{vmatrix} a & b \\ x & y \end{vmatrix} = ay - bx,\ C_{23} = -(ay - bx) = bx - ay$ etc.

Determinant of any order :

Let $A = [a_{ij}]_n$ be a square matrix (n > 1). Determinant of A is defined as the sum of products of elements of any one row (or any one column) with corresponding cofactors.

e.g.1 $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ (using first row).

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$|A| = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$ (using second column).

$$= -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Transpose of a determinant : The transpose of a determinant is the determinant of transpose of the corresponding matrix.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \Rightarrow \quad D^T = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Properties of determinant :

(1) $|A| = |A'|$ for any square matrix A.

i.e. the value of a determinant remains unaltered, if the rows & columns are inter changed,

i.e. $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D'$

(2) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only.

e.g. Let $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D_2 = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D_2 = -D_1$

(3) Let λ be a scalar. Than λ |A| is obtained by multiplying any one row (or any one column) of |A| by λ

$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $E = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then E = KD

(4) $|AB| = |A|\,|B|$.

(5) $|\lambda A| = \lambda^n |A|$, when $A = [a_{ij}]_n$.

(6) A skew-symmetric matrix of odd order has determinant value zero.

(7) If a determinant has all the elements zero in any row or column, then its value is zero,

i.e. $D = \begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

(8) If a determinant has any two rows (or columns) identical (or proportional), then its value is zero,

i.e. $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

(9) If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants, i.e.

$$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(10) The value of a determinant is not altered by adding to the elements of any row (or column) a constant multiple of the corresponding elements of any other row (or column),

i.e. $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D_2 = \begin{vmatrix} a_1 + ma_2 & b_1 + mb_2 & c_1 + mc_2 \\ a_2 & b_2 & c_2 \\ a_3 + na_1 & b_3 + nb_1 & c_3 + nc_1 \end{vmatrix}$. Then $D_2 = D_1$

(11) Let $A = [a_{ij}]_n$. The sum of the products of elements of any row with corresponding cofactors of any other row is zero. (Similarly the sum of the products of elements of any column with corresponding cofactors of any other column is zero).

Example # 9 Simplify $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Solution : Let $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

Apply $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ b-c & c-a & a \\ c-a & a-b & b \end{vmatrix}$$

$= (a + b + c) ((b - c) (a - b) - (c - a)^2)$

$= (a + b + c) (ab + bc - ca - b^2 - c^2 + 2ca - a^2)$

$= (a + b + c) (ab + bc + ca - a^2 - b^2 - c^2) \equiv 3abc - a^3 - b^3 - c^3$

Example # 10 Simplify $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$

Solution : Given determinant is equal to

$$= \frac{1}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & abc & abc \end{vmatrix} = \begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$$

Apply $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$= \begin{vmatrix} a^2-b^2 & b^2-c^2 & c^2 \\ a^3-b^3 & b^3-c^3 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (a-b)\,(b-c)\begin{vmatrix} a+b & b+c & c^2 \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$= (a – b)\ (b – c)\ [ab^2 + abc + ac^2 + b^3 + b^2C + bc^2 – a^2b – a^2c – ab^2 – abc – b^3 – b^2c]$
$= (a – b)\ (b – c)\ [c(ab + bc + ca) – a(ab + bc + ca)]$
$= (a – b)\ (b – c)\ (c – a)\ (ab + bc + ca)$

Self practice problems

(5) Find the value of $\Delta = .\begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix}$

(6) Simplify . $\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$

(7) Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a + b + c)^3$.

(8) Show that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a – b)\ (b – c)\ (c – a)$ by using factor theorem .

Answers : (5) 0 (6) 0

Application of determinants : Following examples of short hand writing large expressions are:

(i) Area of a triangle whose vertices are (x_r, y_r); r = 1, 2, 3 is:

$D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ If D = 0 then the three points are collinear.

(ii) Equation of a straight line passing through (x_1, y_1) & (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(iii) The lines :
$a_1x + b_1y + c_1 = 0$........ (1)
$a_2x + b_2y + c_2 = 0$........ (2)
$a_3x + b_3y + c_3 = 0$........ (3)

are concurrent if, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

Condition for the consistency of three simultaneous linear equations in 2 variables.

(iv) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if:

$abc + 2fgh – af^2 – bg^2 – ch^2 = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Singular & non singular matrix : A square matrix A is said to be singular or non-singular according as |A| is zero or non-zero respectively.

Cofactor matrix & adjoint matrix : Let $A = [a_{ij}]_n$ be a square matrix. The matrix obtained by replacing each element of A by corresponding cofactor is called as cofactor matrix of A, denoted as cofactor A. The transpose of cofactor matrix of A is called as adjoint of A, denoted as adj A.

i.e. if $A = [a_{ij}]_n$

then cofactor $A = [c_{ij}]_n$ when c_{ij} is the cofactor of a_{ij} $\forall$ i & j.

Adj $A = [d_{ij}]_n$ where $d_{ij} = c_{ji}$ $\forall$ i & j.

Properties of cofactor A and adj A :

(a) $A \cdot \text{adj } A = |A|\, I_n = (\text{adj } A)\, A$ where $A = [a_{ij}]_n$.

(b) $|\text{adj } A| = |A|^{n-1}$, where n is order of A. In particular, for 3 × 3 matrix, $|\text{adj } A| = |A|^2$

(c) If A is a symmetric matrix, then adj A are also symmetric matrices.

(d) If A is singular, then adj A is also singular.

Example # 11 : For a 3×3 skew-symmetric matrix A, show that adj A is a symmetric matrix.

Solution : $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ cof $A = \begin{bmatrix} c^2 & -bc & ca \\ -bc & b^2 & -ab \\ ca & -ab & a^2 \end{bmatrix}$

adj $A = (\text{cof } A)' = \begin{bmatrix} c^2 & -bc & ca \\ -bc & b^2 & -ab \\ ca & -ab & a^2 \end{bmatrix}$ which is symmetric.

Inverse of a matrix (reciprocal matrix) :

Let A be a non-singular matrix. Then the matrix $\frac{1}{|A|}$ adj A is the multiplicative inverse of A (we call it inverse of A) and is denoted by A^{-1}. We have $A\,(\text{adj } A) = |A|\, I_n = (\text{adj } A)\, A$

$\Rightarrow$ $A\left(\frac{1}{|A|}\text{adj } A\right) = I_n = \left(\frac{1}{|A|}\text{adj } A\right) A$, for A is non-singular

$\Rightarrow$ $A^{-1} = \frac{1}{|A|}$ adj A.

Remarks :

1. The necessary and sufficient condition for existence of inverse of A is that A is non-singular.

2. A^{-1} is always non-singular.

3. If $A = \text{dia}(a_{11}, a_{22}, \ldots\ldots, a_{nn})$ where $a_{ii} \neq 0$ $\forall$ i, then $A^{-1} = \text{diag}(a_{11}^{-1}, a_{22}^{-1}, \ldots., a_{nn}^{-1})$.

4. $(A^{-1})' = (A')^{-1}$ for any non-singular matrix A. Also adj (A') = (adj $A)'$.

5. $(A^{-1})^{-1} = A$ if A is non-singular.

6. Let k be a non-zero scalar & A be a non-singular matrix. Then $(kA)^{-1} = \frac{1}{k} A^{-1}$.

7. $|A^{-1}| = \frac{1}{|A|}$ for $|A| \neq 0$.

8. Let A be a non-singular matrix. Then $AB = AC \Rightarrow B = C$ & $BA = CA \Rightarrow B = C$.

9. A is non-singular and symmetric $\Rightarrow A^{-1}$ is symmetric.

10. $(AB)^{-1} = B^{-1} A^{-1}$ if A and B are non- singular.

11. In general AB = **0** does not imply A = **0** or B = **0**. But if A is non-singular and AB = **0**, then B = **0**. Similarly B is non-singular and AB = **0** $\Rightarrow$ A = **0**. Therefore, AB = **0** $\Rightarrow$ either both are singular or one of them is **0**.

Example # 12 : If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then verify that A adj A = | A | I. Also find A^{-1}

Solution : We have $|A| = 1(16-9) - 3(4-3) + 3(3-4) = 1 \neq 0$

Now $C_{11} = 7$, $C_{12} = -1$, $C_{13} = -1$, $C_{21} = -3$, $C_{22} = 1$, $C_{23} = 0$, $C_{31} = -3$, $C_{32} = 0$, $C_{33} = 1$

Therefore adj $A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Now $A(\text{adj } A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|.\,I$

Also $A^{-1} = \dfrac{1}{|A|}$ adj $A = \dfrac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Example # 13 : Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = O$, where I is 2 × 2 identity matrix and O is 2 × 2 zero matrix. Using the equation, find A^{-1} .

Solution : We have $A^2 = A.A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

Hence $A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Now $A^2 - 4A + I = 0$

Therefore $A A - 4A = -I$

or $AA(A^{-1}) - 4 A A^{-1} = -I A^{-1}$ (Post multiplying by A^{-1} because $|A| \neq 0$)

or $A (A A^{-1}) - 4I = -A^{-1}$ or $AI - 4I = -A^{-1}$

or $A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Hence $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Example # 14 : For two non-singular matrices A & B, show that adj (AB) = (adj B) (adj A)

Solution : We have $(AB)(\text{adj }(AB)) = |AB|\, I_n = |A|\,|B|\, I_n$

$A^{-1}(AB)(\text{adj }(AB)) = |A|\,|B|\, A^{-1}$

$\Rightarrow$ $B \text{ adj }(AB) = |B| \text{ adj } A$ $\left(\because \quad A^{-1} = \dfrac{1}{|A|} \text{ adj } A\right)$

$\Rightarrow$ $B^{-1} B \text{ adj }(AB) = |B|\, B^{-1} \text{ adj } A$

$\Rightarrow$ $\text{adj }(AB) = (\text{adj}B)(\text{adj } A)$

Self practice problems :

(9) If A is non-singular, show that adj (adj A) = $|A|^{n-2}$ A.

(10) Prove that adj $(A^{-1}) = (\text{adj } A)^{-1}$.

(11) For any square matrix A, show that |adj (adj A) | = $|A|^{(n-1)^2}$.

(12) If A and B are non-singular matrices, show that $(AB)^{-1} = B^{-1} A^{-1}$.

Elementary row transformation of matrix :

The following operations on a matrix are called as elementary row transformations.

(a) Interchanging two rows.

(b) Multiplications of all the elements of row by a nonzero scalar.

(c) Addition of constant multiple of a row to another row.

Note : Similar to above we have elementary column transformations also.

Remarks : Two matrices A & B are said to be equivalent if one is obtained from other using elementary transformations. We write $A \approx B$.

Finding inverse using Elementry operations

(i) Using row transformations :

If A is a matrix such that A^{-1} exists, then to find A^{-1} using elementary row operations,

Step I : Write A = IA and

Step II : Apply a sequence of row operation on A = IA till we get, I = BA.

The matrix B will be inverse of A.

Note : In order to apply a sequence of elementary row operations on the matrix equation X = AB, we will apply these row operations simultaneously on X and on the first matrix A of the product AB on RHS.

(ii) Using column transformations :

If A is a matrix such that A^{-1} exists, then to find A^{-1} using elementary column operations,

Step I : Write A = AI and

Step II : Apply a sequence of column operations on A = AI till we get, I = AB.

The matrix B will be inverse of A.

Note : In order to apply a sequence of elementary column operations on the matrix equation X = AB, we will apply these row operations simultaneously on X and on the second matrix B of the product AB on RHS.

Example # 15 : Obtain the inverse of the matrix A = $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ using elementary operations.

Solution : Write A = IA, i.e. $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}, = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

or $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$ (applying $R_1 \leftrightarrow R_2$)

or $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$ (applying $R_3 \to R_3 - 3R_1$)

or $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$ (applying $R_1 \to R_1 - 2R_2$)

or $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$ (applying $R_3 \to R_3 + 5R_2$)

$$\text{or} \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A \text{ (applying } R_3 \to \frac{1}{2} R_3)$$

$$\text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A \text{ (Applying } R_1 \to R_1 + R_3)$$

$$\text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A \text{ (Applying } R_2 \to R_2 - 2R_3)$$

$$\text{Hence } A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

System of linear equations & matrices : Consider the system

$a_{11}\, x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}\, x_2 + \dots\dots\dots + a_{2n}\, x_n = b_2$

....

$a_{m1}x_1 + a_{m2}x_2 + \dots\dots\dots + a_{mn}x_n = b_n$.

$$\text{Let} \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots & a_{2n} \\ \dots & \dots & \dots\dots & \dots \\ a_{m1} & a_{m2} & \dots\dots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \ \& \ B = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ \dots \\ b_n \end{bmatrix}.$$

Then the above system can be expressed in the matrix form as AX = B.
The system is said to be consistent if it has atleast one solution.

System of linear equations and matrix inverse:

If the above system consist of n equations in n unknowns, then we have AX = B where A is a square matrix.

Results :

(1) If A is non-singular, solution is given by $X = A^{-1}B$.

(2) If A is singular, (adj A) B = 0 and all the columns of A are not proportional, then the system has infinitely many solutions.

(3) If A is singular and (adj A) B $\neq$ 0, then the system has no solution (we say it is inconsistent).

Homogeneous system and matrix inverse :

If the above system is homogeneous, n equations in n unknowns, then in the matrix form it is AX = O.
($\because$ in this case $b_1 = b_2 = \dots\dots\ b_n = 0$), where A is a square matrix.

Results :

(1) If A is non-singular, the system has only the trivial solution (zero solution) X = 0

(2) If A is singular, then the system has infinitely many solutions (including the trivial solution) and hence it has non-trivial solutions.

Example # 16 : Solve the system $\begin{matrix} x+y+z=6 \\ x-y+z=2 \\ 2x+y-z=1 \end{matrix}$ using matrix inverse.

Solution : Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$.

Then the system is AX = B.

|A| = 6. Hence A is non singular.

$$\text{Cofactor } A = \begin{bmatrix} 0 & 3 & 3 \\ 2 & -3 & 1 \\ 2 & 0 & -2 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix}$$

$$X = A^{-1} B = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \quad \text{i.e.} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \Rightarrow \quad x = 1,\ y = 2,\ z = 3.$$

Self practice problems:

(13) $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$. Find the inverse of A using |A| and adj A.

(14) Find real values of λ and μ so that the following systems has
(i) unique solution (ii) infinitely many solutions (iii) No solution.
x + y + z = 6
x + 2y + 3z = 1
x + 2y + λz = μ

(15) Find λ so that the following homogeneous system have a non zero solution
x + 2y + 3z = λx
3x + y + 2z = λy
2x + 3y + z = λz

Answers : (13) $\begin{vmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{vmatrix}$ (14) (i) $\lambda \neq 3$, $\mu \in R$ (ii) $\lambda = 3$, $\mu = 1$ (iii) $\lambda = 3$, $\mu \neq 1$ (15) $\lambda = 6$

Exercise-1

✍ **Marked questions are recommended for Revision.**
✍ **चिन्हित प्रश्न दोहराने योग्य प्रश्न है।**

PART - I : SUBJECTIVE QUESTIONS
भाग - I : विषयात्मक प्रश्न (SUBJECTIVE QUESTIONS)

Section (A): Matrix, Algebra of Matrix, Transpose, symmetric and skew symmetric matrix,
खण्ड (A) : आव्यूह, आव्यूह का बीजगणित, परिवर्त, सममित एवं विषम सममित आव्यूह,

A-1. Construct a 3 × 2 matrix whose elements are given by $a_{ij} = 2i - j$.
एक 3 × 2 मैट्रिक्स बनाइए जिसके अवयव $a_{ij} = 2i - j$ से दिये जाते है।

Ans. $\begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 5 & 4 \end{bmatrix}$

Sol. $a_{ij} = 2i - j$
$a_{11} = 2 - 1 = 1,$ $a_{21} = 4 - 1 = 3,$ $a_{31} = 2 \times 3 - 1 = 5$
$a_{12} = 2 - 2 = 0,$ $a_{22} = 2.2 - 2 = 2,$ $4_{32} = 2 \times 3 - 2 = 4.$

$\therefore A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 5 & 4 \end{bmatrix}$

A-2. If $\begin{bmatrix} x-y & 1 & z \\ 2x-y & 0 & w \end{bmatrix} = \begin{bmatrix} -1 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix}$, find x, y, z, w.
यदि $\begin{bmatrix} x-y & 1 & z \\ 2x-y & 0 & w \end{bmatrix} = \begin{bmatrix} -1 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ हो, तो x, y, z, w ज्ञात कीजिए।

Ans. (x, y, z, w) = (1, 2, 4, 5)

Sol. $\begin{bmatrix} x-y & 1 & z \\ 2x-y & 0 & w \end{bmatrix} = \begin{bmatrix} -1 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix}$
on comparison (तुलना करने पर)
$x - y = -1 \quad \Rightarrow \quad 2x - y = 0 \quad \Rightarrow \quad z = 4 \quad \Rightarrow \quad w = 5$
Hence (अत:) x = 1, y = 2, z = 4, w = 5

A-3. Let $A + B + C = \begin{bmatrix} 4 & -1 \\ 0 & 1 \end{bmatrix}$, $4A + 2B + C = \begin{bmatrix} 0 & -1 \\ -3 & 2 \end{bmatrix}$ and $9A + 3B + C = \begin{bmatrix} 0 & -2 \\ 2 & 1 \end{bmatrix}$ then find A
माना $A + B + C = \begin{bmatrix} 4 & -1 \\ 0 & 1 \end{bmatrix}$, $4A + 2B + C = \begin{bmatrix} 0 & -1 \\ -3 & 2 \end{bmatrix}$ और $9A + 3B + C = \begin{bmatrix} 0 & -2 \\ 2 & 1 \end{bmatrix}$ तब A ज्ञात कीजिए –

Ans. $\begin{bmatrix} 2 & -1/2 \\ 4 & -1 \end{bmatrix}$

Sol. $3A + B = \begin{bmatrix} -4 & 0 \\ -3 & -1 \end{bmatrix}$, $5A + B = \begin{bmatrix} 0 & -1 \\ 5 & -1 \end{bmatrix}$

$\Rightarrow 2A = \begin{bmatrix} 4 & -1 \\ 8 & -2 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 2 & -1/2 \\ 4 & -1 \end{bmatrix}$

A-4. If $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 & 6 \\ 7 & -8 & 2 \end{bmatrix}$, will AB be equal to BA. Also find AB & BA.

यदि $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \\ 5 & 6 \end{bmatrix}$ और $B = \begin{bmatrix} 4 & 5 & 6 \\ 7 & -8 & 2 \end{bmatrix}$ हो, तो क्या BA और AB बराबर होंगे ? AB और BA भी ज्ञात कीजिए।

Ans. $AB = \begin{bmatrix} 18 & -11 & 10 \\ -16 & 47 & 10 \\ 62 & -23 & 42 \end{bmatrix}$, $BA = \begin{bmatrix} 49 & 24 \\ -7 & 58 \end{bmatrix}$

Sol. $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \\ 5 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 5 & 6 \\ 7 & -8 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 4+14 & 5-16 & 6+4 \\ 12-28 & 15+32 & 18-8 \\ 20+42 & 25-48 & 30+12 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 18 & -11 & 10 \\ -16 & 47 & 10 \\ 62 & -23 & 42 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4+15+30 & 8-20+36 \\ 7-24+10 & 14+32+12 \end{bmatrix} \Rightarrow BA = \begin{bmatrix} 49 & 24 \\ -7 & 58 \end{bmatrix}$$

A-5. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then show that $A^3 = \begin{bmatrix} 7 & -12 \\ 3 & -5 \end{bmatrix}$

यदि $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ हो, तो प्रदर्शित कीजिए कि $A^3 = \begin{bmatrix} 7 & -12 \\ 3 & -5 \end{bmatrix}$

Sol. Method विधी -1

$$A^2 = A.A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$$

$$\Rightarrow \quad A^3 = A^2 . A = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \Rightarrow \quad A^3 = \begin{bmatrix} 7 & -12 \\ 3 & -5 \end{bmatrix}$$

Method विधी – 2 (mathematical Induction गणितीय आगमन सिद्धान्त से)

$$A^1 = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

Let माना $A^m = \begin{bmatrix} 1+2m & -4m \\ m & 1-2m \end{bmatrix} \Rightarrow A^{m+1} = \begin{bmatrix} 1+2(m+1) & -4(m+1) \\ m+1 & 1-2(m+1) \end{bmatrix}$

Also भी $A^{m+1} = A^m . A = \begin{bmatrix} 1+2m & -4m \\ m & 1-2m \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3+6m-4m & -4-8m+4m \\ 3m+1-2m & -4m-1+2m \end{bmatrix}$

$$= \begin{bmatrix} 1+2(m+1) & -4(m+1) \\ m+1 & 1-2(m+1) \end{bmatrix}$$

If A^m is true then A^{m+1} is also true so A^m is true in general
यदि A^m सत्य है तो A^{m+1} भी सत्य है अतः सामान्यतः A^m भी सत्य है।

A-6. If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ show that $(I + A) = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

यदि $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ हो, तो प्रदर्शित कीजिए कि $(I + A) = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

Sol. $(I + A) = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$\text{L.H.S.} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix}$$

$$\text{R.H.S.} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}\right)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & +\tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cos\alpha + \sin\alpha\tan\frac{\alpha}{2} & -\sin\alpha + \tan\frac{\alpha}{2}\cos\alpha \\ -\tan\frac{\alpha}{2}\cos\alpha + \sin\alpha & \sin\alpha\tan\frac{\alpha}{2} + \cos\alpha \end{bmatrix} \Rightarrow \begin{bmatrix} \cos\alpha + 2\sin\frac{\alpha}{2}\sin\frac{\alpha}{2} & \sin\frac{\alpha}{2}\left(-2\cos\frac{\alpha}{2} + \frac{\cos\alpha}{\cos\frac{\alpha}{2}}\right) \\ \sin\frac{\alpha}{2}\left(-\frac{\cos\alpha}{\cos\frac{\alpha}{2}} + 2\cos\frac{\alpha}{2}\right) & \cos\alpha + 2\sin^2\frac{\alpha}{2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix}$$

A-7. Given $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$. If $x \in R$ Then for what values of y, $F(x + y) = F(x) F(y)$.

दिया गया हैं कि $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$. यदि $x \in R$ हो, तो y के किस मान के लिए $F(x + y) = F(x) F(y)$ है।

Ans. $y \in R$

Sol. $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $F(x + y) = \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$F(x).\ F(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos x\cos y - \sin x\sin y & -\cos x\sin y - \sin x\cos y & 0 \\ \sin x\cos y + \cos x\sin y & \cos x\cos y - \sin x\sin y & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix} = F\ (x + y)$$

$\therefore y \in R$

A-8. Let $A = [a_{ij}]_{n \times n}$ where $a_{ij} = i^2 - j^2$. Show that A is skew-symmetric matrix.

माना $A = [a_{ij}]_{n \times n}$ जहाँ $a_{ij} = i^2 - j^2$, प्रदर्शित कीजिए A विषम सममित हैं।

Sol. $A = \begin{vmatrix} 1^2-1^2 & 1^2-2^2 & 1^2-3^2 \\ 2^2-1^2 & 2^2-2^2 & 2^2-3^2 \\ 3^2-1^2 & 3^2-2^2 & 3^2-3^2 \end{vmatrix} = \begin{vmatrix} 0 & -3 & -8 \\ 3 & 0 & -5 \\ 8 & 5 & 0 \end{vmatrix}$

So A is skew symmetric matrix

अतः A विषम सममित आव्यूह है

A-9. If $C = \begin{bmatrix} 1 & 4 & 6 \\ 7 & 2 & 5 \\ 9 & 8 & 3 \end{bmatrix}\begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 7 & 9 \\ 4 & 2 & 8 \\ 6 & 5 & 3 \end{bmatrix}$, then trace of $C + C^3 + C^5 + \ldots\ldots + C^{99}$ is

यदि $C = \begin{bmatrix} 1 & 4 & 6 \\ 7 & 2 & 5 \\ 9 & 8 & 3 \end{bmatrix}\begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 7 & 9 \\ 4 & 2 & 8 \\ 6 & 5 & 3 \end{bmatrix}$ हो तब $C + C^3 + C^5 + \ldots\ldots + C^{99}$ का अनुरेख है –

Ans. Zero

Sol. $C = ABA^T$ where $B^T = -B$

$\Rightarrow C^T = (A^T)^T B^T A^T = -ABA^T = -C$

$\Rightarrow$ C is skew matrix $\Rightarrow C^3, C^5, \ldots\ldots\ldots, C^{99}$ are also skew matrix

$\Rightarrow$ trace of $C + C^3 + C^5 + \ldots\ldots + C^{99}$ is zero

Hindi. $C = ABA^T$ जहाँ $B^T = -B$

$\Rightarrow C^T = (A^T)^T B^T A^T = -ABA^T = -C$

$\Rightarrow$ C विषम सममित है $\Rightarrow C^3, C^5, \ldots\ldots\ldots, C^{99}$ विषम सममित है

$\Rightarrow C + C^3 + C^5 + \ldots\ldots + C^{99}$ का अनुरेख शून्य है

Section (B) : Determinant of Matrix
खण्ड (B) : आव्यूह का सारणिक

B-1. If the minor of three-one element (i.e. M_{31}) in the determinant $\begin{vmatrix} 0 & 1 & \sec\alpha \\ \tan\alpha & -\sec\alpha & \tan\alpha \\ 1 & 0 & 1 \end{vmatrix}$ is 1 then find the value of α. $(0 \le \alpha \le \pi)$.

दी गई सारणिक $\begin{vmatrix} 0 & 1 & \sec\alpha \\ \tan\alpha & -\sec\alpha & \tan\alpha \\ 1 & 0 & 1 \end{vmatrix}$ में अवयव 3-1 के उपसारणिक (अर्थात M_{31}) का मान 1 है, तो α का मान ज्ञात कीजिये जहाँ $(0 \le \alpha \le \pi)$.

Ans. $0, \frac{3\pi}{4}, \pi$

Sol. $\begin{vmatrix} 0 & 1 & \sec\alpha \\ \tan\alpha & -\sec\alpha & \tan\alpha \\ 1 & 0 & 1 \end{vmatrix}$; $M_{31} = \begin{vmatrix} 1 & \sec\alpha \\ -\sec\alpha & \tan\alpha \end{vmatrix} = 1$

$\tan\alpha + \sec^2\alpha = 1$ $(\because 1 + \tan^2\alpha = \sec^2\alpha)$

$\tan\alpha = -\tan^2\alpha \Rightarrow \tan\alpha(1 + \tan\alpha) = 0 \Rightarrow \tan\alpha = 0, \tan\alpha = -1$

$\alpha = n\pi$ or या $\alpha = n\pi - \frac{\pi}{4}$, $n \in I$

B-2. Using the properties of determinants, evalulate:

सारणिक के गुणधर्मो के उपयोग से निम्न के मान ज्ञात कीजिये।

(i) $\begin{vmatrix} 23 & 6 & 11 \\ 36 & 5 & 26 \\ 63 & 13 & 37 \end{vmatrix}$ (ii) $\begin{vmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{vmatrix}$

(iii) $\begin{vmatrix} 103 & 115 & 114 \\ 111 & 108 & 106 \\ 104 & 113 & 116 \end{vmatrix} + \begin{vmatrix} 113 & 116 & 104 \\ 108 & 106 & 111 \\ 115 & 114 & 103 \end{vmatrix}$.

(iv) $\begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix}$

Ans. (i) 0 (ii) 0 (iii) 0 (iv) $5(3\sqrt{2}-5\sqrt{3})$

Sol. (i) $\begin{vmatrix} 23 & 6 & 11 \\ 36 & 5 & 26 \\ 63 & 13 & 37 \end{vmatrix}$

Applying $C_2 \to 2C_2 + C_3$ से

$\begin{vmatrix} 23 & 23 & 11 \\ 36 & 36 & 11 \\ 63 & 63 & 37 \end{vmatrix}$

In this two rows are similar so |A| = 0

इसमें दो पंक्तियों समान है इसलिए |A| = 0

(ii) $\begin{vmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{vmatrix}$

It is skew-symmetric determinant of odd order so |A| = 0

यह विषम क्रम की विषम सममित आव्यूह है इसलिए |A| = 0

(iii) $\begin{vmatrix} 103 & 115 & 114 \\ 111 & 108 & 106 \\ 104 & 113 & 116 \end{vmatrix} + \begin{vmatrix} 113 & 116 & 104 \\ 108 & 106 & 111 \\ 115 & 114 & 103 \end{vmatrix}$

Applying operation in first determinant

प्रथम सारणिक में

$C_1 \leftrightarrow C_2$

$-\begin{vmatrix} 115 & 103 & 114 \\ 108 & 111 & 106 \\ 113 & 104 & 116 \end{vmatrix} + \begin{vmatrix} 113 & 116 & 104 \\ 108 & 106 & 111 \\ 115 & 114 & 103 \end{vmatrix}$

Applying operation in first determinant प्रथम सारणिक

$C_2 \leftrightarrow C_3$

$= \begin{vmatrix} 115 & 114 & 103 \\ 108 & 106 & 111 \\ 113 & 116 & 104 \end{vmatrix} + \begin{vmatrix} 113 & 116 & 104 \\ 108 & 106 & 111 \\ 115 & 114 & 103 \end{vmatrix}$

Applying operation in first determinant प्रथम सारणिक

$R_1 \leftrightarrow R_3$

$-\begin{vmatrix} 113 & 116 & 104 \\ 108 & 106 & 111 \\ 115 & 114 & 103 \end{vmatrix} + \begin{vmatrix} 113 & 116 & 104 \\ 108 & 106 & 111 \\ 115 & 114 & 103 \end{vmatrix} = 0$

(iv) $\begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix} = \begin{vmatrix} \sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15} & 5 & \sqrt{10} \\ 3 & \sqrt{15} & 5 \end{vmatrix} + \begin{vmatrix} \sqrt{13} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{26} & 5 & \sqrt{10} \\ \sqrt{65} & \sqrt{15} & 5 \end{vmatrix}$

$= \sqrt{3}\sqrt{5}\sqrt{5}\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{5} & \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{5} \end{vmatrix} + \sqrt{13}\sqrt{5}\sqrt{5}\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{2} & \sqrt{5} & \sqrt{2} \\ \sqrt{5} & \sqrt{3} & \sqrt{5} \end{vmatrix}$

$C_1 \to C_1 - C_2$

$$= 5\sqrt{3}\begin{vmatrix} -1 & 2 & 1 \\ 0 & \sqrt{5} & \sqrt{2} \\ 0 & \sqrt{3} & \sqrt{5} \end{vmatrix} = 5(3\sqrt{2} - 5\sqrt{3})$$

B-3. Prove that :

सिद्ध कीजिए –

(i) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)(a + b + c)$

(ii) $\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix} = -(a + b + c)(a - b)(b - c)(c - a)$

(iii) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

(iv) If $\begin{vmatrix} 1 & a^2 & a^4 \\ 1 & b^2 & b^4 \\ 1 & c^2 & c^4 \end{vmatrix} = (a + b)(b + c)(c + a)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$.

यदि $\begin{vmatrix} 1 & a^2 & a^4 \\ 1 & b^2 & b^4 \\ 1 & c^2 & c^4 \end{vmatrix} = (a + b)(b + c)(c + a)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$.

Sol. (i) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)(a + b + c)$

Applying $C_1 \to C_1 - C_2$ & $C_2 \to C_2 - C_3$ से

$$\begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix} = (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+b^2+ab & b^2+c^2+bc & c^3 \end{vmatrix}$$

$= (a - b)(b - c)(c - a)(a + b + c)$

(ii) $\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}$

Applying $C_2 \to C_2 + C_1$ (संक्रिया $C_2 \to C_2 + C_1$ से)

$$\begin{vmatrix} a & a+b+c & a^2 \\ b & b+c+a & b^2 \\ c & c+a+b & c^2 \end{vmatrix} = (a+b+c)\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix}$$

$$= -(a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = -(a + b + c)(a - b)(b - c)(c - a)$$

(iii) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$ by $R_1 \to R_1 + R_2 + R_3$

$= 2\begin{vmatrix} b+c & a+c & a+b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$ by $R_1 \to R_1 - R_3$

$= 2\begin{vmatrix} b & a & 0 \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 2\,[b((c + a)(a + b) - cb) - a(ab + b^2 - bc)]$

$= 2\,[b(ac + cb + a^2 + ab - cb) - a^2b - ab^2 + abc]$

$= 2\,[abc + a^2b + ab^2 - a^2b - ab^2 + abc] = 4\,abc$

(iv) $\begin{vmatrix} 1 & a^2 & a^4 \\ 1 & b^2 & b^4 \\ 1 & c^2 & c^4 \end{vmatrix} = K\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \Rightarrow (a^2 - b^2)(b^2 - c^2)(c^2 - a^2) = K(a - b)(b - c)(c - a)$

$\therefore \quad K = (a + b)(b + c)(c + a)$

B-4. If a, b, c are positive and are the p^{th}, q^{th}, r^{th} terms respectively of a G.P., show without expanding that,

$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0.$

यदि गुणोत्तर श्रेढ़ी के pवें, qवें तथा r वें पद क्रमशः a, b, c तथा धनात्मक हो, तो बिना विस्तार के प्रदर्शित कीजिए कि

$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0.$

Sol. $\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0$

Here $a = AR^{p-1}$, $b = AR^{q-1}$, $C = AR^{r-1}$

$\begin{vmatrix} \log AR^{p-1} & p & 1 \\ \log AR^{q-1} & q & 1 \\ \log AR^{r-1} & r & 1 \end{vmatrix} = \begin{vmatrix} \log A + (p-1)\log R & p & 1 \\ \log A + (q-1)\log R & q & 1 \\ \log A + (r-1)\log R & r & 1 \end{vmatrix}$

Applying $R_2 \to R_2 \to R_1$ & $R_3 \to R_3 - R_1$

$\begin{vmatrix} \log A + (p-1)\log R & p & 1 \\ \log R(q-p) & q-p & 0 \\ \log R(r-p) & r-p & 0 \end{vmatrix} = (r - p)(q - p)\begin{vmatrix} \log A + (p-1)\log R & p & 1 \\ \log R & 1 & 0 \\ \log R & 1 & 0 \end{vmatrix} = 0$

B-5. Find the non – zero roots of the equation,

निम्नलिखित समीकरणों के अशून्य मूल ज्ञात कीजिये–

(i) $\Delta = \begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ ax+b & bx+c & c \end{vmatrix} = 0.$ (ii) $\begin{vmatrix} 15-2x & 11 & 10 \\ 11-3x & 17 & 16 \\ 7-x & 14 & 13 \end{vmatrix} = 0$

Ans. (i) $x = -2\,b/a$ (ii) 4

Sol. (i) $\begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ ax+b & bx+c & c \end{vmatrix} = 0$

Applying संक्रिया $C_3 \to C_3 - C_2$ से

$\begin{vmatrix} a & b & ax \\ b & c & bx \\ ax+b & bx+c & -bx \end{vmatrix} = 0 \Rightarrow x \begin{vmatrix} a & b & a \\ b & c & b \\ ax+b & bx+c & -b \end{vmatrix} = 0$

Applying संक्रिया $C_1 \to C_1 - C_3$ से

$x \begin{vmatrix} 0 & b & a \\ 0 & c & b \\ ax+2b & bx+c & -b \end{vmatrix} \Rightarrow x(ax + 2b)(b^2 - ac) = 0$

$\therefore$ Non zero root of equation समीकरण का अशून्य मूल $x = -\dfrac{2b}{a}$

(ii) $\begin{vmatrix} 15-2x & 11 & 10 \\ 11-3x & 17 & 16 \\ 7-x & 14 & 13 \end{vmatrix}$

Applying संक्रिया $C_2 \to C_2 - C_3$ से

$\begin{vmatrix} 15-2x & 1 & 10 \\ 11-3x & 1 & 16 \\ 7-x & 1 & 13 \end{vmatrix} = 0$

Applying संक्रिया $R_1 \to R_1 - R_3$ & एवं $R_2 \to R_2 - R_3$ से

$\begin{vmatrix} 8-x & 0 & -3 \\ 4-2x & 0 & 3 \\ 7-x & 1 & 13 \end{vmatrix} = 0 \Rightarrow -1[(8 - x)3 + 3(4 - 2x)] = 0 \Rightarrow 9x = 36 \Rightarrow x = 4$

B-6. If $S_r = \alpha^r + \beta^r + \gamma^r$ then show that $\begin{vmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{vmatrix} = (\alpha - \beta)^2 (\beta - \gamma)^2 (\gamma - \alpha)^2$.

यदि $S_r = \alpha^r + \beta^r + \gamma^r$ हो, तो प्रदर्शित कीजिए कि $\begin{vmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{vmatrix} = (\alpha - \beta)^2 (\beta - \gamma)^2 (\gamma - \alpha)^2$.

Sol. $\Delta = \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} = \begin{vmatrix} 3 & \alpha+\beta+\gamma & \alpha^2+\beta^2+\gamma^2 \\ \alpha+\beta+\gamma & \alpha^2+\beta^2+\gamma^2 & \alpha^3+\beta^3+\gamma^3 \\ \alpha^2+\beta^2+\gamma^2 & \alpha^3+\beta^3+\gamma^3 & \alpha^4+\beta^4+\gamma^4 \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix}^2 = (\alpha - \beta)^2 (\beta - \gamma)^2 (\gamma - \alpha)^2$

B-7. Show that $\begin{vmatrix} a_1 l_1 + b_1 m_1 & a_1 l_2 + b_1 m_2 & a_1 l_3 + b_1 m_3 \\ a_2 l_1 + b_2 m_1 & a_2 l_2 + b_2 m_2 & a_2 l_3 + b_2 m_3 \\ a_3 l_1 + b_3 m_1 & a_3 l_2 + b_3 m_2 & a_3 l_3 + b_3 m_3 \end{vmatrix} = 0.$

प्रदर्शित कीजिए कि $\begin{vmatrix} a_1 l_1 + b_1 m_1 & a_1 l_2 + b_1 m_2 & a_1 l_3 + b_1 m_3 \\ a_2 l_1 + b_2 m_1 & a_2 l_2 + b_2 m_2 & a_2 l_3 + b_2 m_3 \\ a_3 l_1 + b_3 m_1 & a_3 l_2 + b_3 m_2 & a_3 l_3 + b_3 m_3 \end{vmatrix} = 0.$

Sol. $\begin{vmatrix} a_1\ell_1+b_1m_1 & a_1\ell_2+b_1m_2 & a_1\ell_3+b_1m_3 \\ a_2\ell_1+b_2m_1 & a_2\ell_2+b_2m_2 & a_2\ell_3+b_2m_3 \\ a_3\ell_1+b_3m_1 & a_3\ell_2+b_3m_2 & a_3\ell_3+b_3m_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix}\begin{vmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ 0 & 0 & 0 \end{vmatrix} = 0$

B-8. If $\begin{vmatrix} e^x & \sin x \\ \cos x & \ell n(1+x) \end{vmatrix} = A + Bx + Cx^2 + ...$, then find the value of A and B.

यदि $\begin{vmatrix} e^x & \sin x \\ \cos x & \ell n(1+x) \end{vmatrix} = A + Bx + Cx^2 + ...$, तब A तथा B का मान ज्ञात कीजिए।

Ans. A = 0, B = 0

Sol. $\begin{vmatrix} e^x & \sin x \\ \cos x & \ln(1+x) \end{vmatrix} = A + Bx + Cx^2 + Dx^3 + ...$

Putting x = 0 $\Rightarrow$ A = 0.
Further on differentiating w.r.t. x

$\begin{vmatrix} e^x & \sin x \\ -\sin x & \ln(1+x) \end{vmatrix} + \begin{vmatrix} e^x & \cos x \\ \cos x & \dfrac{1}{(1+x)} \end{vmatrix} = B + 2Cx + 3Dx^2 + ...$

Putting x = 0 $\Rightarrow$ B = 0

Hindi. $\begin{vmatrix} e^x & \sin x \\ \cos x & \ln(1+x) \end{vmatrix} = A + Bx + Cx^2 + Dx^3 + ...$

x = 0 रखने पर $\Rightarrow$ A = 0.
x तके सापेक्ष अवकलन करने पर

$\begin{vmatrix} e^x & \sin x \\ -\sin x & \ln(1+x) \end{vmatrix} + \begin{vmatrix} e^x & \cos x \\ \cos x & \dfrac{1}{(1+x)} \end{vmatrix} = B + 2Cx + 3Dx^2 + ...$

x = 0 रखने पर $\Rightarrow$ B = 0

Section (C) : Cofactor matrix, adj matrix and inverse of matrix
खण्ड (C) : सहखण्ड आव्यूह, सहखण्डज आव्यूह एवं आव्यूह का प्रतिलोम

C-1. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$ and AB – CD = 0 find D.

यदि $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$ और AB – CD = 0 हो, तो D ज्ञात कीजिए।

Ans. $\begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$

Sol. $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$

Let माना $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then हो तो $AB = \begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix}$ & एवं $CD = \begin{bmatrix} 3a+5c & 2b+5d \\ 3a+8c & 3b+8d \end{bmatrix}$

$AB - CD = \begin{bmatrix} 3-2a-5c & -2b-5d \\ 43-3a-8c & 22-3b-8d \end{bmatrix} = 0$

$\therefore$ a = – 191, c = 77, b = –110, d = 44

$D = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$

C-2. (i)✍ Prove that (adj adj A) = $|A|^{n-2}$ A

(ii) Find the value of |adj adj adj A| in terms of |A|

(i) सिद्ध कीजिए कि (adj adj A) = $|A|^{n-2}$ A

(ii) |adj adj adj A| का |A| के पदों में मान ज्ञात कीजिए।

Ans. (ii) $|A|^{(n-1)^3}$

Sol. (i) we know that A. adj A = |A| . I and $|\text{adj } A| = |A|^{n-1}$

$\therefore$ adj A. (adj adj A) = |adj A| I

Pre-multiplying A $\Rightarrow$ A. adj A. (adj adj A) = A $|A|^{n-1}$ I

$\Rightarrow$ |A| . I (adj adj A) = A $|A|^{(n-1)}$ $\Rightarrow$ (adj adj A) = A $|A|^{(n-2)}$

(ii) We know that $|\text{adj } A| = |A|^{n-1}$ $\Rightarrow$ $|\text{adj }(\text{adj}(\text{adj } A))| = |\text{adj }(\text{adj } A)|^{n-1}$

$= |\text{adj}A|^{(n-1)^2} = |A|^{(n-1)^3}$

Hindi. (i) हम जानते हैं कि A. adj A = |A| . I और $|\text{adj } A| = |A|^{n-1}$

$\therefore$ adj A. (adj adj A) = |adj A| I

A से पूर्व गुणन करने पर

$\Rightarrow$ A. adj A. (adj adj A) = A $|A|^{n-1}$ I $\Rightarrow$ |A| . I (adj adj A) = A $|A|^{(n-1)}$

$\Rightarrow$ (adj adj A) = A $|A|^{(n-2)}$

(ii) हम जानते हैं कि $|\text{adj } A| = |A|^{n-1}$

$|\text{adj }(\text{adj}(\text{adj } A))| = |\text{adj }(\text{adj } A)|^{n-1} = |\text{adj}A|^{(n-1)^2} = |A|^{(n-1)^3}$

C-3. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$

यदि $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ और $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ हो, तो $(AB)^{-1}$ ज्ञात कीजिए।

Ans. $\begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Sol. $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$(AB)^{-1} = B^{-1}A^{-1}$

$\therefore$ Matrix formed by Cofactors of B = $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}$

Transpose of formed by Cofactors of B = adjB = $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

$\therefore$ $B^{-1} = \dfrac{\text{adj}B}{|B|} \Rightarrow B^{-1} = \dfrac{\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}}{1}$ $\therefore$ $B^{-1}A^{-1} = \begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Hindi. $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$(AB)^{-1} = B^{-1}A^{-1}$ $\qquad \therefore$ B के सहखण्डों से बना आव्यूह $B = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}$

B के सहखण्डों को परिवर्तित करके B = adjB = बनाते है

$\therefore \qquad B^{-1} = \dfrac{\text{adj}B}{|B|} \Rightarrow B^{-1} = \dfrac{\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}}{1} \qquad \therefore \qquad B^{-1}A^{-1} = \begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

C-4. If A is a symmetric and B skew symmetric matrix and (A + B) is non-singular and $C = (A + B)^{-1} (A - B)$, then prove that
(i) $C^T (A + B) C = A + B$ (ii) $C^T (A - B) C = A - B$

यदि A सममित है तथा B विषम सममित आव्यूह है तथा (A + B) व्युत्क्रमणीय है और $C = (A + B)^{-1} (A - B)$, तब सिद्ध कीजिये
(i) $C^T (A + B) C = A + B$ (ii) $C^T (A - B) C = A - B$

Sol. (i) $(A + B) C = (A + B) (A + B)^{-1} (A - B)$
$(A + B) C = A - B$ ....(i)
$C^T = ((A + B)^{-1} (A - B))^T = (A - B)^T ((A + B)^{-1})^T$
$= (A - B)^T ((A + B)^T)^{-1}$ Here $\{|A + B| \neq 0 \Rightarrow |(A + B)^T| \neq 0 \Rightarrow |A - B| \neq 0\}$
$= (A + B) (A - B)^{-1}$ ....(ii)
By (i) and (ii) (i) व (ii) से
$C^T (A + B) C = (A + B) (A - B)^{-1} (A - B)$
$C^{-1} (A + B) C = A + B$ ....(iii)
(ii) Taking transpose of (iii) का परिवर्त लेने पर
$C^T (A - B) C = A - B.$

C-5. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & c \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$, then find values of a & c.

$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & c \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$ हो तो a और c का मान ज्ञात कीजिए।

Ans. a = 1, c = – 1

Sol. $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix}$ $\qquad A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & c \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$

$AA^{-1} = I \Rightarrow \frac{1}{2}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & 2c \\ 5 & -3 & 1 \end{bmatrix} = I \Rightarrow \frac{1}{2}\begin{bmatrix} 2 & 0 & 2c+2 \\ 0 & 2 & 4c+4 \\ 8-8a & 6a-6 & 4+2ca \end{bmatrix} \Rightarrow = I$

$\therefore \qquad 2c + 2 = 0, 8 - 8a = 0 \quad \Rightarrow \quad c = -1, a = 1$

Section (D) : Charactristic equation and system of equations
खण्ड (D) : अभिलाक्षणिक समीकरण एवं समीकरणों का निकाय

D-1. For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ find a & b so that $A^2 + aA + bI = 0$. Hence find A^{-1} .

मैट्रिक्स $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ के लिए a और b के मान ज्ञात कीजिए जबकि $A^2 + aA + bI = 0$ हो तथा A^{-1} भी ज्ञात कीजिए।

Ans. $a = -4,\ b = 1,\ A^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

Sol. $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$

$A^2 + aA + bI = 0 \Rightarrow \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 3a & 2a \\ a & a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 11+3a+b & 8+2a \\ 4+a & 3+a+b \end{bmatrix} = 0$

$8 + 2a = 0,\ 3 + a + b = 0 \Rightarrow a = -4,\ b = 1$

$A^{-1} = \dfrac{\text{adj}\ A}{|A|} = \dfrac{\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}}{1} \Rightarrow A^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

D-2. Find the total number of possible square matrix A of order 3 with all real entries, whose adjoint matrix B has characterstics polynomial equation as $\lambda^3 - \lambda^2 + \lambda + 1 = 0$.

3 क्रम के सभी वर्ग आव्यूह A की संख्या ज्ञात कीजिए जिसके सभी अवयव वास्तविक है तथा जिसकी सहखण्डज आव्यूह B की अभिलाक्षणिक समीकरण $\lambda^3 - \lambda^2 + \lambda + 1 = 0$ है।

Ans. 0

Sol. $|B| = -1 \Rightarrow |A|^2 = -1 \Rightarrow$ A has not all real entries के वास्तविक अवयव नही है।

D-3. If $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$, show that $A^3 = (5A - I)(A - I)$

यदि $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ हो, तो प्रदर्शित कीजिए कि $A^3 = (5A - I)(A - I)$

Sol. $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$; $A^3 = 5A^2 - 6AI + I^2$

$A^2 = \begin{bmatrix} 3 & 3 & 7 \\ 1 & 4 & 4 \\ 3 & 1 & 6 \end{bmatrix}$ and और $A^3 = \begin{bmatrix} 10 & 9 & 23 \\ 5 & 9 & 14 \\ 9 & 5 & 19 \end{bmatrix}$

$5A^2 - 6A + I = \begin{bmatrix} 10 & 9 & 23 \\ 5 & 9 & 14 \\ 9 & 5 & 19 \end{bmatrix} = A^3$

D-4. Apply Cramer's rule to solve the following simultaneous equations.

क्रेमर नियम के प्रयोग से निम्नलिखित समीकरण निकाय का हल ज्ञात कीजिए।

(i) $2x + y + 6z = 46$
$5x - 6y + 4z = 15$
$7x + 4y - 3z = 19$

(ii) $x + 2y + 3z = 2$
$x - y + z = 3$
$5x - 11y + z = 17$

Ans. (i) $x = 3,\ y = 4,\ z = 6$

(ii) $x=-\frac{5k}{3}+\frac{8}{3}$, $y=-\frac{2k}{3}-\frac{1}{3}$, z = k, where जहाँ $k \in R$

Sol. (i) $2x + y + 6z = 46$
$5x - 6y + 4z = 15$
$7x + 4y - 3z = 19$

$$\Delta = \begin{vmatrix} 2 & 1 & 6 \\ 5 & -6 & 4 \\ 7 & 4 & -3 \end{vmatrix} = 419,\ \Delta_x = \begin{vmatrix} 46 & 1 & 6 \\ 15 & -6 & 4 \\ 19 & 4 & -3 \end{vmatrix} = 1257,$$

$$\Delta y = \begin{vmatrix} 2 & 46 & 6 \\ 5 & 15 & 4 \\ 7 & 19 & -3 \end{vmatrix} = 1676,\ \Delta_z = \begin{vmatrix} 2 & 1 & 46 \\ 5 & -6 & 15 \\ 7 & 4 & 19 \end{vmatrix} = 2514$$

$$x = \frac{\Delta_x}{\Delta} = 3,\ y = \frac{\Delta_y}{\Delta} = 4,\ z = \frac{\Delta_z}{\Delta} = 6$$

(ii) $x + 2y + 3z = 2$
$x - y + z = 3$
$5x - 11y + z = 17$

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 5 & -11 & 1 \end{vmatrix} = 0,\ \Delta_x = \begin{vmatrix} 2 & 2 & 3 \\ 3 & -1 & 1 \\ 17 & -11 & 1 \end{vmatrix} = 0,\ \Delta_y = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 5 & 17 & 1 \end{vmatrix} = 0,\ \Delta_z = \begin{vmatrix} 1 & 2 & 2 \\ 1 & -1 & 3 \\ 5 & -11 & 17 \end{vmatrix} = 0$$

$\therefore$ given system of equations has infinite solutions.
दी गई समीकरण निकाय के अनन्त हल है

Let माना z = k

$x + 2y = 2 - 3k$
$x - y = 3 - k$

$\therefore$ $x=-\frac{5k}{3}+\frac{8}{3}$, $y=-\frac{2k}{3}-\frac{1}{3}$, z = k, where जहाँ $k \in R$

D-5. Solve using Cramer's rule: $\frac{4}{x+5}+\frac{3}{y+7} = -1$ & $\frac{6}{x+5}-\frac{6}{y+7} = -5$.

क्रेमर नियम के प्रयोग से हल कीजिए : $\frac{4}{x+5}+\frac{3}{y+7} = -1$ तथा $\frac{6}{x+5}-\frac{6}{y+7} = -5$.

Ans. $x = -7,\ y = -4$

Sol. $\frac{4}{x+5}+\frac{3}{y+7} = -1$ & $\frac{6}{x+5}-\frac{6}{y+7} = 5$

Put $\frac{1}{x+5} = t$ & $\frac{1}{y+7} = \mu$ रखने पर

$4t + 3\mu = -1$ & $6t - 6\mu = -5 \Rightarrow t = \frac{-1}{2}$ & $\mu = 1/3 \Rightarrow x = -7,\ y = -4$

D-6. Find those values of c for which the equations:
$2x+3y = 3$
$(c+2)x + (c+4)y = c+6$
$(c+2)^2x + (c+4)^2y = (c+6)^2$ are consistent.
Also solve above equations for these values of c.
c के वह मान ज्ञात कीजिए जिनके लिए समीकरण निकाय
$2x+3y = 3$
$(c+2)x + (c+4)y = c+6$
$(c+2)^2x + (c+4)^2y = (c+6)^2$ संगत है

तथा c के इन मानों के लिए ऊपर दी गई समीकरणों को हल कीजिए।

Ans. for c = 0, x = –3, y = 3 के लिए; for c = –10 के लिए, $x = -\frac{1}{2}$, $y = \frac{4}{3}$

Sol. For consistency of the equation $\begin{vmatrix} 2 & 3 & 3 \\ c+2 & c+4 & c+6 \\ (c+2)^2 & (c+4)^2 & (c+6)^2 \end{vmatrix} = 0$

this gives c = 0 and c = – 10
for c = 0, we get
$x = -3$
$y = 3$ and for c = – 10, we get $x = -\frac{1}{2}$, $y = \frac{4}{3}$

Hindi समीकरणों की संगत्ता के लिए $\begin{vmatrix} 2 & 3 & 3 \\ c+2 & c+4 & c+6 \\ (c+2)^2 & (c+4)^2 & (c+6)^2 \end{vmatrix} = 0$

अतः c = 0 और c = – 10
c = 0 के लिए–
$x = -3$
$y = 3$ और c = – 10 के लिए $x = -\frac{1}{2}$, $y = \frac{4}{3}$

D-7. Solve the following systems of linear equations by matrix method.
निम्न रेखिक समीकरण निकाय को आव्यूह विधि की सहायता से हल कीजिए–

(i) $2x - y + 3z = 8$
$-x + 2y + z = 4$
$3x + y - 4z = 0$

(ii) $x + y + z = 9$
$2x + 5y + 7z = 52$
$2x + y - z = 0$

Ans. **(i)** x = 2, y = 2, z = 2 **(ii)** x = 1, y = 3, z = 5

Sol. **(i)** $2x - y + 3z = 8$
$-x + 2y + z = 4$
$3x + y - 4z = 0$
AX = B

$$\Rightarrow \quad X = A^{-1}B \quad \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad |A| = -38 \quad \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{38} \begin{bmatrix} 9 & 1 & 7 \\ 1 & 17 & 5 \\ 7 & 5 & -3 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix} = \frac{1}{38} \begin{bmatrix} 76 \\ 76 \\ 76 \end{bmatrix}$$

$\Rightarrow$ x = 2, y = 2 & z = 2

(ii) $x + y + z = 9$
$2x + 5y + 7z = 52$
$2x + y - z = 0$

$$\Rightarrow \quad X = A^{-1}B \quad \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad |A| = -4 \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -1/2 & -1/2 \\ -4 & 3/4 & 5/4 \\ 2 & 1/4 & -3/4 \end{bmatrix} \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 31 \end{bmatrix} \Rightarrow x = 1, y = 3 \ \& \ z = 31$$

D-8. Investigate for what values of λ, μ the simultaneous equations
$x + y + z = 6;\ x + 2y + 3z = 10\ \&\ x + 2y + \lambda z = \mu$ have;
(a) A unique solution
(b) An infinite number of solutions.
(c) No solution.
λ तथा μ के वह मान ज्ञात कीजिए जिनके लिए समीकरण निकाय
$x + y + z = 6;\ x + 2y + 3z = 10$ एवं $x + 2y + \lambda z = \mu$ के
(a) अद्वितीय हल होगें। (b) अनन्त हल हैं। (c) कोई हल नहीं हैं।
Ans. **(a)** $\lambda \neq 3$ **(b)** $\lambda = 3, \mu = 10$ **(c)** $\lambda = 3, \mu \neq 10$

Sol. $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} = \lambda - 3$; $\Delta_x = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ \mu & 2 & \lambda \end{vmatrix} = 2\lambda + \mu - 16$

$\Delta_y = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 10 & 3 \\ 1 & \mu & \lambda \end{vmatrix} = 4\lambda - 2\mu + 8$; $\Delta_z = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 10 \\ 1 & 2 & \mu \end{vmatrix} = (\mu - 10)$

(i) For unique solution $\Delta \neq 0 \therefore \lambda \neq 3$
(ii) For infinite no. of solution $\Delta = 0, \Delta_x = 0, \Delta_y = 0, \Delta_z = 0$.
$\therefore \lambda = 3, \mu = 10$
(iii) For no. solution $\Delta = 0$, & at least are of $\Delta_x, \Delta_y, \Delta_z$ is non-zero.
$\therefore \lambda = 3, \mu \neq 10$

Hindi. $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} = \lambda - 3$; $\Delta_x = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ \mu & 2 & \lambda \end{vmatrix} = 2\lambda + \mu - 16$

$\Delta_y = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 10 & 3 \\ 1 & \mu & \lambda \end{vmatrix} = 4\lambda - 2\mu + 8$; $\Delta_z = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 10 \\ 1 & 2 & \mu \end{vmatrix} = (\mu - 10)$

(i) अद्वितीय हल के लिए $\Delta \neq 0 \therefore \lambda \neq 3$
(ii) अनन्त हलों की संख्या के लिए $\Delta = 0, \Delta_x = 0, \Delta_y = 0, \Delta_z = 0$.
$\therefore \lambda = 3, \mu = 10$
(iii) हलों की संख्या के लिए $\Delta = 0$, तथा $\Delta_x, \Delta_y, \Delta_z$ मे से कम से कम एक अशून्य है
$\therefore \lambda = 3, \mu \neq 10$

D-9. Determine the product $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ and use it to solve the system of equations $x - y + z = 4$, $x - 2y - 2z = 9$, $2x + y + 3z = 1$.
गुणनफल $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ ज्ञात कीजिए और इसका उपयोग करके समीकरण निकाय
$x - y + z = 4$, $x - 2y - 2z = 9$, $2x + y + 3z = 1$ को हल कीजिए।
Ans. $x = 3, y = -2, z = -1$

Sol. $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} \Rightarrow AB = 8I$

$x - y + z = 4 \Rightarrow x - 2y - 2z = 9 \Rightarrow 2x + y + 3z = 1$

$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} \Rightarrow BX = C$

$\because \quad AB = 8I \quad \Rightarrow \quad \frac{1}{8}AB = I$

$B^{-1}B = I \quad \Rightarrow \quad B^{-1} = \frac{A}{8}$

So (अतः) $X = B^{-1}C$

$$X = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} \Rightarrow \quad x = 3, y = -2, z = -1$$

D-10. Compute A^{-1}, if $A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$. Hence solve the matrix equations $\begin{bmatrix} 3 & 0 & 3 \\ 2 & 1 & 0 \\ 4 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 2y \\ z \\ 3y \end{bmatrix}$.

यदि $A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$ हो, तो A^{-1} ज्ञात कीजिए। आव्यूह समीकरण $\begin{bmatrix} 3 & 0 & 3 \\ 2 & 1 & 0 \\ 4 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 2y \\ z \\ 3y \end{bmatrix}$ को हल कीजिए।

Ans. $x = 1, y = 2, z = 3, \; A^{-1} = \frac{1}{17}\begin{bmatrix} 1 & 5 & 1 \\ 8 & 6 & -9 \\ 10 & -1 & -7 \end{bmatrix}$

Sol. $A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$

$$\text{Adj.}A = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix} \Rightarrow \quad |A| = -17 \quad \Rightarrow \quad A^{-1} = \frac{\text{Adj}A}{|A|}$$

$$A^{-1} = \frac{1}{17}\begin{bmatrix} 1 & 5 & 1 \\ 8 & 6 & -9 \\ 10 & -1 & -7 \end{bmatrix} \Rightarrow \begin{bmatrix} 3x+0+3z \\ 2x+y+0 \\ 4x+0+2z \end{bmatrix} = \begin{bmatrix} 8+2y \\ 1+z \\ 4+3y \end{bmatrix}$$

$\Rightarrow \quad 3x + 3z = 8 + 2y \Rightarrow \quad 2x + y = 1 + z \quad \Rightarrow \quad 4x + 2z = 4 + 3y$

By Solving हल करने पर $x = 1, y = 2$ & $z = 3$

D-11. Which of the following statement(s) is/are true निम्न मे से कौनसा कथन सत्य है:

S1 : The system of equations $\begin{matrix} 4x-5y-2z=2 \\ 5x-4y+2z=3 \\ 2x+2y+8z=1 \end{matrix}$ is Inconsistent.

समीकरण निकाय $\begin{matrix} 4x-5y-2z=2 \\ 5x-4y+2z=3 \\ 2x+2y+8z=1 \end{matrix}$ असंगत है।

S2 : A matrix 'A' has 6 elements. The number of possible orders of A is 6.

एक मैट्रिक्स A में 6 अवयव हैं, तब A के सभी संभव क्रमों (orders) की संख्या 6 है।

S3 : For any 2 × 2 matrix A, if A (adjA) = $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, then |A| = 10.

किसी 2 × 2 क्रम के मैट्रिक्स के लिए यदि A (adjA) = $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ हो, तो |A| = 10.

S4 : If A is skew symmetric, then B′AB is also skew symmetric.
यदि A विषम सममित है, तो B′AB भी विषम सममित होगा।

Ans. **S1, S3, S4**

Sol. **S1** $\Delta = \begin{vmatrix} 4 & -5 & -2 \\ 5 & -4 & 2 \\ 2 & 2 & 8 \end{vmatrix}$

A = 0, singular
And (adj A) B = 0 system is inconsistent

S2 Obviously the possible orders are 1 × 6, 2 × 3, 3 × 2 and 6 × 1.
No. of possible orders is 4

S3 A (adj A) = $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ $\Rightarrow$ A (adj A) = 10 $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $\Rightarrow$ A (adj A) = $|A|I_n$

$\therefore$ $|A|$ = 10

S4 if C = B′AB
C′ = (B′ AB)′ = B′A (B′)′ = B′A′B = – B′AB here A′ = – A

Hindi. **S1** $\Delta = \begin{vmatrix} 4 & -5 & -2 \\ 5 & -4 & 2 \\ 2 & 2 & 8 \end{vmatrix}$

A = 0, अव्युत्क्रमणीय है
तथा (adj A) B = 0 असंगत निकाय है

S2 स्पष्टतः संभव क्रम 1 × 6, 2 × 3, 3 × 2 तथा 6 × 1 है
संभावित क्रमों की संख्या 4 है

S3 A (adj A) = $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ $\Rightarrow$ A (adj A) = 10 $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $\Rightarrow$ A (adj A) = $|A|I_n$

$\therefore$ $|A|$ = 10

S4 यदि C = B′AB
C′ = (B′ AB)′ = B′A (B′)′ = B′A′B = – B′AB यहाँ A′ = – A

D-12✍_. Let A is non scalar matrix of order two satisfying the relation $A^2 + \ell A + mI = O$, then prove that characterstic equation of matrix A is $x^2 + \ell x + m = 0$.

माना A दो क्रम का अशून्य अदिश आव्यूह है जो सम्बन्ध $A^2 + \ell A + mI = O$ को संतुष्ठ करता है तब सिद्ध कीजिए कि आव्यूह A का अभिलाक्षणिक समीकरण $x^2 + \ell x + m = 0$ है।

Sol. Let A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Now $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \ell\begin{bmatrix} a & b \\ c & d \end{bmatrix} + m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ $a^2 + bc + \ell a + m = 0,\ ab + bd + \ell b = 0,$

$ac + dc + \ell c = 0,\ bc + d^2 + \ell d + m = 0$

Now $b(a + d + \ell) = 0,\ c(a + d + \ell) = 0$

Because A is non scalar, so atleast one of the b, c is non zero, so $a + d + \ell = 0 \Rightarrow \ell = -(a + d)$

Now $a^2 + bc + a(-a - d) + m = 0 \Rightarrow bc - ad + m = 0$ Þ $m = ad - bc$

Now characterstic equation is $x^2 - (a + d)x + ad - bc = 0 \Rightarrow x^2 + \ell x + m = 0$

Sol. माना $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

अब $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \ell\begin{bmatrix} a & b \\ c & d \end{bmatrix} + m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow a^2 + bc + \ell a + m = 0,\ ab + bd + \ell b = 0,$

$ac + dc + \ell c = 0,\ bc + d^2 + \ell d + m = 0$

अब $b(a + d + \ell) = 0,\ c(a + d + \ell) = 0$

चूंकि A अदिश नहीं है इसलिए b, c मे से कम से कम एक अशून्य है $a + d + \ell = 0 \Rightarrow \ell = -(a + d)$

अब $a^2 + bc + a(-a - d) + m = 0 \Rightarrow bc - ad + m = 0 \Rightarrow m = ad - bc$

अब अभिलाक्षणिक समीकरण $x^2 - (a + d)x + ad - bc = 0 \Rightarrow x^2 + \ell x + m = 0$

PART - II : ONLY ONE OPTION CORRECT TYPE

भाग - II : केवल एक सही विकल्प प्रकार (ONLY ONE OPTION CORRECT TYPE)

Section (A): Matrix, Algebra of Matrix, Transpose, symmetric and skew symmetric matrix,

खण्ड (A) : आव्यूह, आव्यूह का बीजगणित, परिवर्त, सममित एवं विषम सममित आव्यूह,

A-1. $\begin{bmatrix} x^2+x & x \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -x+1 & x \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 5 & 1 \end{bmatrix}$ then x is equal to -

(A*) – 1 (B) 2 (C) 1 (D) No value of x

$\begin{bmatrix} x^2+x & x \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -x+1 & x \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 5 & 1 \end{bmatrix}$ तो x का मान है –

(A) – 1 (B) 2 (C) 1 (D) x का कोई मान नहीं

Sol. $\begin{bmatrix} x^2+x & x \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -x+1 & x \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 5 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x^2+x & x-1 \\ -x+4 & x+2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 5 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x^2+x & x-1 \\ -x+4 & x+2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 5 & 1 \end{bmatrix}$

On comparing तुलना करने पर

$x^2 + x = 0 \Rightarrow x = 0, -1$ $x - 1 = -2 \Rightarrow x = -1$

$-x + 4 = 5 \Rightarrow x = -1$ $x + 2 = 1 \Rightarrow x = -1$

Hence the value of x is – 1. अतः x का मान –1 है

A-2. If A = $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and B = $\begin{bmatrix}-5 & 4 & 0\\0 & 2 & -1\\1 & -3 & 2\end{bmatrix}$, then

(A) AB = $\begin{bmatrix}-5 & 8 & 0\\0 & 4 & -2\\3 & -9 & 6\end{bmatrix}$ (B) AB = [– 2 – 1 4]

(C) AB = $\begin{bmatrix}-1\\1\\1\end{bmatrix}$ (D*) AB does not exist

यदि A = $\begin{bmatrix}1\\2\\3\end{bmatrix}$ और B = $\begin{bmatrix}-5 & 4 & 0\\0 & 2 & -1\\1 & -3 & 2\end{bmatrix}$ हो, तो –

(A) AB = $\begin{bmatrix}-5 & 8 & 0\\0 & 4 & -2\\3 & -9 & 6\end{bmatrix}$ (B) AB = [– 2 – 1 4]

(C) AB = $\begin{bmatrix}-1\\1\\1\end{bmatrix}$ (D) AB विद्यमान नहीं है।

Sol. Matrix A has order (3 × 1) and Matrix B has order (3 × 3).
So multiplication AB is not possible.

Hindi. आव्यूह A का क्रम (3 × 1) तथा आव्यूह B का क्रम (3 × 3) है
अतः AB का गुणनफल संभव नहीं है

A-3. If AB = O for the matrices

A = $\begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{bmatrix}$ and B = $\begin{bmatrix}\cos^2\phi & \cos\phi\sin\phi\\ \cos\phi\sin\phi & \sin^2\phi\end{bmatrix}$ then $\theta - \phi$ is

(A*) an odd multiple of $\frac{\pi}{2}$ (B) an odd multiple of π

(C) an even multiple of $\frac{\pi}{2}$ (D) 0

यदि A = $\begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{bmatrix}$ और B = $\begin{bmatrix}\cos^2\phi & \cos\phi\sin\phi\\ \cos\phi\sin\phi & \sin^2\phi\end{bmatrix}$ के लिए AB = O हो, तो $\theta - \phi$ हैं –

(A) $\frac{\pi}{2}$ का विषम गुणज (B) π का विषम गुणज (C) $\frac{\pi}{2}$ का सम गुणज (D) 0

Sol. AB = 0

$$\Rightarrow \begin{bmatrix}\cos^2\theta\cos^2\phi + \sin\theta\sin\phi\cos\theta\cos\phi & \cos^2\theta\cos\phi\sin\phi + \sin^2\phi\cos\theta\sin\theta\\ \cos^2\phi\cos\theta\sin\theta + \sin^2\theta\cos\phi\sin\phi & \cos\theta\sin\theta\cos\phi\sin\phi + \sin^2\theta\sin^2\phi\end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix}\cos\theta\cos\phi\cos(\theta-\phi) & \cos\theta\sin\phi(\cos\theta-\phi)\\ \cos\phi\sin\theta\cos(\theta-\phi) & \sin\theta\sin\phi\cos(\theta-\phi)\end{bmatrix} = 0 \Rightarrow \cos(\theta-\phi)\begin{bmatrix}\cos\theta\cos\phi & \cos\theta\sin\phi\\ \cos\phi\sin\theta & \sin\theta\sin\phi\end{bmatrix} = 0$$

$\Rightarrow \cos(\theta - \phi) = 0 \Rightarrow \theta - \phi = (2n + 1)\frac{\pi}{2}$

A-4. If I = $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, J = $\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$ and B = $\begin{bmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{bmatrix}$, then B =

यदि I = $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, J = $\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$ और B = $\begin{bmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{bmatrix}$, हो, तो B =

(A*) I$\cos\theta$ + J$\sin\theta$ (B) I$\cos\theta$ – J$\sin\theta$ (C) I$\sin\theta$ + J$\cos\theta$ (D) – I$\cos\theta$ + J$\sin\theta$

Sol. $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$I \cos\theta + J \sin\theta = \begin{bmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{bmatrix} + \begin{bmatrix} 0 & \sin\theta \\ -\sin\theta & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

A-5. In an upper triangular matrix $A = [a_{ij}]_{n \times n}$ the elements $a_{ij} = 0$ for
(A) $i < j$ (B) $i = j$ (C*) $i > j$ (D) $i \le j$
एक ऊपरी त्रिभुजाकार मैट्रिक्स $A = [a_{ij}]_{n \times n}$ में अवयव $a_{ij} = 0$ होता है –
(A) $i < j$ के लिये (B) $i = j$ के लिये (C) $i > j$ के लिये (D) $i \le j$ के लिये

Sol. For upper triangle matrix, elements below diagonal are zero
$\therefore \quad a_{ij} = 0$, where $i > j$

Hindi. ऊपरी त्रिभुजाकार मैट्रिक्स में विकर्ण के नीचे के अवयव शून्य होते है।
$\therefore \quad a_{ij} = 0$, जहाँ $i > j$

A-6. If A = diag (2, –1, 3), B = diag (–1, 3, 2), then A^2B =
यदि A = diag (2, –1, 3), B = diag (–1, 3, 2) हो, तो A^2B =
(A) diag (5, 4, 11) (B*) diag (– 4, 3, 18) (C) diag (3, 1, 8) (D) B

Sol. A = diag (2, –1, 3), B = diag (–1, 3, 2) then (हो तो) $A^2 B$ = ?

$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$; $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$A^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}$; $A^2B = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 18 \end{bmatrix}$ = diag (–4, 3, 18)

A-7. If A is a skew- symmetric matrix, then trace of A is
(A) 1 (B) – 1 (C*) 0 (D) none of these
यदि A एक विषम सममित मैट्रिक्स हो, तो A का अनुरेख (trace) हैं –
(A) 1 (B) – 1 (C) 0 (D) इनमें से कोई नहीं

Sol. Trace of A = $a_{11} + a_{22} + a_{33}$
For skew symmetric matrix $a_{11} = a_{22} = a_{33} = 0$
Trace of A = 0

Hindi. A का अनुरेख = $a_{11} + a_{22} + a_{33}$
विषम सममित मैट्रिक्स के लिए $a_{11} = a_{22} = a_{33} = 0$
A का अनुरेख = 0

A-8. Let $A = \begin{bmatrix} p & q \\ q & p \end{bmatrix}$ such that det(A) = r where p, q, r all prime numbers, then trace of A is equal to
माना $A = \begin{bmatrix} p & q \\ q & p \end{bmatrix}$ इस प्रकार है की det(A) = r जहाँ p, q, r अभाज्य संख्यायें है, तो A का अनुरेख बराबर है–
(A*) 6 (B) 5 (C) 2 (D) 3

Sol. $p^2 - q^2 = r$ p = 3 q = 2, r = 5

A-9. $A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ and तथा $(A^8 + A^6 + A^4 + A^2 + I)\,V = \begin{bmatrix} 31 \\ 62 \end{bmatrix}$.
(Where I is the (2 × 2) identity matrix), then the product of all elements of matrix V is
(जहाँ I, (2 × 2) क्रम का इकाई आव्यूह है), तब आव्यूह V के सभी अवयवों के गुणनफल है
(A*) 2 (B) 1 (C) 3 (D) –2

Sol. $A^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$A^4 = A^2 \cdot A^2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 2^2 & 0 \\ 0 & 2^2 \end{bmatrix}$, $A^6 = \begin{bmatrix} 2^3 & 0 \\ 0 & 2^3 \end{bmatrix}$, $A^8 = \begin{bmatrix} 2^4 & 0 \\ 0 & 2^4 \end{bmatrix}$

$\therefore \quad (A^8 + A^6 + A^4 + A^2 + I)\, V = \begin{bmatrix} 31 \\ 62 \end{bmatrix} \Rightarrow \begin{bmatrix} 31 & 0 \\ 0 & 31 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 31 \\ 62 \end{bmatrix} \Rightarrow \quad x = 1, y = 2$

$\therefore \quad xy = 2.$

A-10. Let $A = \begin{bmatrix} 3x^2 \\ 1 \\ 6x \end{bmatrix}$, $B = [a\ b\ c]$ and $C = \begin{bmatrix} (x+2)^2 & 5x^2 & 2x \\ 5x^2 & 2x & (x+1)^2 \\ 2x & (x+2)^2 & 5x^2 \end{bmatrix}$

Where a, b, c and $x \in R$, Given that tr (AB) = tr(C), then the value of (a + b + c).

मानाकि $A = \begin{bmatrix} 3x^2 \\ 1 \\ 6x \end{bmatrix}$, $B = [a\ b\ c]$ तथा $C = \begin{bmatrix} (x+2)^2 & 5x^2 & 2x \\ 5x^2 & 2x & (x+1)^2 \\ 2x & (x+2)^2 & 5x^2 \end{bmatrix}$

जहाँ a, b, c तथा $x \in R$, दिया गया है कि tr (AB) = tr(C), तब (a + b + c) का मान है

(A*) 7 (B) 2 (C) 1 (D) 4

Sol. $AB = \begin{bmatrix} 3ax^2 & 3bx^2 & 3cx^2 \\ a & b & c \\ 6ax & 6bx & 6cx \end{bmatrix}$

Now अब, $\quad tr\,(AB) = tr\,(C) \Rightarrow 3ax^2 + b + 6cx = (x+2)^2 + 2x + 5x^2$

$3ax^2 + 6c + b = 6x^2 + 6x + 4 \Rightarrow a = 2, b = 4, c = 1 \Rightarrow a + b + c = 7.$

A-11. Which one of the following is Correct ?

(i) The elements on the main diagonal of a symmetric matrix are all zero

(ii) The elements on the main diagonal of a skew - symmetric matrix are all zero

(iii) For any square matrix A, A A′ is symmetric

(iii) For any square matrix A, $(A + A')^2 = A^2 + (A')^2 + 2AA'$

(A*) FFTF (B) FFTF (B) (A*) FTTT (B) TFFT

निम्न में से कौनसा कथन सही हैं –

(A*) एक सममित मैट्रिक्स के मुख्य विकर्ण के सभी अवयव शून्य होते हैं।

(B) एक विषम सममित मैट्रिक्स के मुख्य विकर्ण के सभी अवयव शून्य होते हैं।

(C) किसी वर्ग मैट्रिक्स A के लिए, A A′ सममित हैं।

(D*) किसी वर्ग मैट्रिक्स A के लिए, $(A + A')^2 = A^2 + (A')^2 + 2AA'$ हैं।

Sol. The elements of main diagonal of skew symmetric matrix are all zero but not necessarily for symmetric matrix.

Hindi. विषम सममित आव्यूह के मुख्य विकर्ण के अवयव सभी शून्य होते है। परन्तु सममित आव्यूह के लिये यह आवश्यक नही है।

Section (B) : Determinant of Matrix
खण्ड (B) : आव्यूह का सारणिक

B-1. If A and B are square matrices of order 3 such that |A| = – 1, |B| = 3, then |3AB| is equal to

यदि A और B , 3 क्रम की वर्ग मैट्रिक्स इस प्रकार है कि |A| = – 1, |B| = 3 हो, तो |3AB| का मान हैं –

(A) – 9 (B*) – 81 (C) – 27 (D) 81

Sol. $|3\,AB| = |A|\,.\,|3B|_{3\times3} = (-1)\,.\,3^3\,|B| = -81$

B-2. The absolute value of the determinant $\begin{vmatrix} -1 & 2 & 1 \\ 3+2\sqrt{2} & 2+2\sqrt{2} & 1 \\ 3-2\sqrt{2} & 2-2\sqrt{2} & 1 \end{vmatrix}$ is:

(A*) $16\sqrt{2}$ (B) $8\sqrt{2}$ (C) 8 (D) none

सारणिक $\begin{vmatrix} -1 & 2 & 1 \\ 3+2\sqrt{2} & 2+2\sqrt{2} & 1 \\ 3-2\sqrt{2} & 2-2\sqrt{2} & 1 \end{vmatrix}$ का निरपेक्ष मान है–

(A*) $16\sqrt{2}$ (B) $8\sqrt{2}$ (C) 8 (D) इनमें से कोई नहीं

Sol. $\begin{bmatrix} -1 & 2 & 1 \\ 3+2\sqrt{2} & 2+2\sqrt{2} & 1 \\ 3-2\sqrt{2} & 2-2\sqrt{2} & 1 \end{bmatrix}$

Applying संक्रिया $R_2 \rightarrow R_2 - R_1$ & $R_3 \rightarrow R_3 - R_1$

$= \begin{bmatrix} -1 & 2 & 1 \\ 4+2\sqrt{2} & 2\sqrt{2} & 0 \\ 4-2\sqrt{2} & -2\sqrt{2} & 0 \end{bmatrix} = 1(-8\sqrt{2} - 8 - 8\sqrt{2} + 8) = -16\sqrt{2}$

So absolute value is $-16\sqrt{2}$ अतः निरपेक्ष मान $-16\sqrt{2}$ है।

B-3. If α, β & γ are the roots of the equation $x^3 + px + q = 0$ then the value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} =$

(A) p (B) q (C) $p^2 - 2q$ (D*) none

यदि α, β और γ समीकरण $x^3 + px + q = 0$ के मूल हो, तो सारणिक $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ का मान है –

(A) p (B) q (C) $p^2 - 2q$ (D) इनमें से कोई नहीं

Sol. α, β, γ are roots of $x^3 + px + q = 0$

$\therefore$ $\alpha + \beta + \gamma = 0$ Here $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$

$(\alpha + \beta + \gamma) \begin{vmatrix} 1 & \beta & \gamma \\ 1 & \gamma & \alpha \\ 1 & \alpha & \beta \end{vmatrix} = 0$

Hindi. $x^3 + px + q = 0$ के मूल α, β, γ है

$\therefore$ $\alpha + \beta + \gamma = 0$ यहाँ $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$

संक्रिया $C_1 \rightarrow C_1 + C_2 + C_3$ से

$(\alpha + \beta + \gamma) \begin{vmatrix} 1 & \beta & \gamma \\ 1 & \gamma & \alpha \\ 1 & \alpha & \beta \end{vmatrix} = 0$

B-4. If a, b, c > 0 & x, y, z ∈ R, then the determinant $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^y+b^{-y})^2 & (b^y-b^{-y})^2 & 1 \\ (c^z+c^{-z})^2 & (c^z-c^{-z})^2 & 1 \end{vmatrix}$ =

(A) $a^xb^yc^z$ (B) $a^{-x}b^{-y}c^{-z}$ (C) $a^{2x}b^{2y}c^{2z}$ (D*) zero

यदि a, b, c > 0 और x, y, z ∈ R हो, तो सारणिक $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^y+b^{-y})^2 & (b^y-b^{-y})^2 & 1 \\ (c^z+c^{-z})^2 & (c^z-c^{-z})^2 & 1 \end{vmatrix}$ =

(A) $a^xb^yc^z$ (B) $a^{-x}b^{-y}c^{-z}$ (C) $a^{2x}b^{2y}c^{2z}$ (D) शून्य

Sol. $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^y+b^{-y})^2 & (b^3-b^{-y})^2 & 1 \\ (c^z+c^{-z})^2 & (c^z-c^{-z})^2 & 1 \end{vmatrix}$

Applying संक्रिया $C_1 \to C_1 - C_2$

$\begin{vmatrix} 4 & (a^x-a^{-x})^2 & 1 \\ 4 & (b^y-b^{-y})^2 & 1 \\ 4 & (c^z-c^{-z})^2 & 1 \end{vmatrix} = 0$

B-5. If a, b & c are non-zero real numbers, then D = $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$ =

(A) abc (B) $a^2b^2c^2$ (C) bc + ca + ab (D*) zero

यदि a, b और c अशून्य वास्तविक संख्याएँ हो, तो D = $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$ =

(A) abc (B) $a^2b^2c^2$ (C) bc + ca + ab (D) शून्य

Sol. $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = a^2b^2c^2 \begin{vmatrix} bc & 1 & \frac{1}{c}+\frac{1}{b} \\ ca & 1 & \frac{1}{a}+\frac{1}{c} \\ ab & 1 & \frac{1}{b}+\frac{1}{a} \end{vmatrix} = abc \begin{vmatrix} bc & 1 & ab+ac \\ ca & 1 & bc+ab \\ ab & 1 & ac+bc \end{vmatrix} = abc\ (ab + bc + ca)$

$\begin{vmatrix} bc & 1 & 1 \\ ca & 1 & 1 \\ ab & 1 & 1 \end{vmatrix} = 0$

$C_3 \to C_3 + C_1$

B-6. The determinant $\begin{vmatrix} b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \\ b_3+c_3 & c_3+a_3 & a_3+b_3 \end{vmatrix}$ =

(A) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (B*) 2 $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (C) 3 $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (D) none of these

सारणिक $\begin{vmatrix} b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \\ b_3+c_3 & c_3+a_3 & a_3+b_3 \end{vmatrix}$ =

(A) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (B*) 2 $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (C) 3 $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (D) इनमें से कोई नहीं

Sol. $\begin{vmatrix} b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+c_2 & a_2+b_2 \\ b_3+c_3 & c_3+a_3 & a_3+b_3 \end{vmatrix}$

Taking two common, applying $C_1 \to C_1 + C_2 + C_3$ (संक्रिया $C_1 \to C_1 + C_2 + C_3$ से दो उभयनिष्ठ लेने पर)

$$= \begin{vmatrix} 2(a_1+b_1+c_1) & c_1+a_1 & a_1+b_1 \\ 2(a_2+b_2+c_2) & c_2+a_2 & a_2+b_2 \\ 2(a_3+b_3+c_3) & c_3+a_3 & a_3+b_3 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ & $C_3 \to C_3 - C_1$ (संक्रिया $C_2 \to C_2 - C_1$ एवं $C_3 \to C_3 - C_1$ से)

$$= 2\begin{vmatrix} a_1+b_1+c_1 & -b_1 & -c_1 \\ a_2+b_2+c_2 & -b_2 & -c_2 \\ a_3+b_3+c_3 & -b_3 & -c_3 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$ (संक्रिया $C_1 \to C_1 + C_2 + C_3$ से)

$$= 2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

B-7. If x, y, z $\in$ R & $\Delta = \begin{vmatrix} x & x+y & x+y+z \\ 2x & 5x+2y & 7x+5y+2z \\ 3x & 7x+3y & 9x+7y+3z \end{vmatrix} = -16$ then value of x is

यदि x, y, z $\in$ R तथा $\Delta = \begin{vmatrix} x & x+y & x+y+z \\ 2x & 5x+2y & 7x+5y+2z \\ 3x & 7x+3y & 9x+7y+3z \end{vmatrix} = -16$ हो, तो x का मान है –

(A) – 2 (B) – 3 (C*) 2 (D) 3

Sol. $\Delta = \begin{vmatrix} x & x+y & x+y+z \\ 2x & 5x+2y & 7x+5y+2z \\ 3x & 7x+3y & 9x+7y+3z \end{vmatrix} = -16$

Applying $R_2 \to R_2 - 2R_1$ & $R_3 \to R_3 - 3R_1$ (संक्रिया $R_2 \to R_2 - 2R_1$ एव $R_3 \to R_3 - 3R_1$ से)

$$\Delta = \begin{vmatrix} x & x+y & x+y+z \\ 0 & 3x & 5x+3y \\ 0 & 4x & 6x+4y \end{vmatrix} = -16$$

Applying $R_3 \to R_3 - R_2$ (संक्रिया $R_3 \to R_3 - R_2$ से)

$$\begin{vmatrix} x & x+y & x+y+z \\ 0 & 3x & 5x+3y \\ 0 & x & x+y \end{vmatrix} = -16$$

Applying $R_2 \to R_2 - 3R_1$ (संक्रिया $R_2 \to R_2 - 3R_1$ से)

$$\begin{vmatrix} x & x+y & x+y+z \\ 0 & 0 & 2x \\ 0 & x & x+y \end{vmatrix} = -16 \Rightarrow -2x(x^2 - 0) = -16 \Rightarrow x^3 = 8 \Rightarrow x = 2$$

B-8. The determinant $\begin{vmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & \cos 2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix}$ is:

(A) 0 (B*) independent of θ (C) independent of ϕ (D) independent of θ & ϕ both

सारणिक $\begin{vmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & \cos 2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix}$ का मान है–

(A) 0 (B) θ से स्वतन्त्र (C) ϕ से स्वतन्त्र (D) θ और ϕ दोनों से स्वतन्त्र

Sol. $\begin{vmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & \cos 2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix} = \dfrac{1}{\sin\phi\cos\phi}\begin{vmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & \cos 2\phi \\ \sin\theta\sin\phi & \sin\phi\cos\theta & \sin^2\phi \\ -\cos\theta\cos\phi & \sin\theta\cos\phi & \cos^2\phi \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$ (संक्रिया $R_1 \to R_1 + R_2 + R_3$ से)

$$= \frac{1}{\sin\phi\cos\phi}\begin{vmatrix} 0 & 0 & 2\cos^2\phi \\ \sin\theta\sin\phi & \sin\phi\cos\theta & \sin^2\phi \\ -\cos\theta\cos\phi & \sin\theta\cos\phi & \cos^2\phi \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 2\cos^2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix} = 2\cos^2\phi\,(\sin^2\theta + \cos^2\theta) = 2\cos^2\phi$$

B-9s. Let A be set of all determinants of order 3 with entries 0 or 1, B be the subset of A consisting of all determinants with value 1 and C be the subset of A consisting of all determinants with value –1. Then
STATEMENT -1 : The number of elements in set B is equal to number of elements in set C.
and
STATEMENT-2 : $(B \cap C) \subset A$

(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B*) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

मानाकि A कोटि 3 के सभी सारणिकों जिनके अवयव 0 या 1 है, का समुच्चय है। समुच्चय A के उन सभी सारणिकों का उपसमुच्चय B है जिन सारणिकों का मान 1 है। समुच्चय A के उन सभी सारणिकों का उपसमुच्चय C है जिन सारणिकों का मान –1 है। तब
वक्तव्य-1 : समुच्चय B में अवयवों की संख्या समुच्चय C में अवयवों की संख्या के बराबर है।
और

वक्तव्य-2 : $(B \cap C) \subset A$

(A) कथन–1 सत्य है, कथन–2 सत्य है ; कथन–2, कथन–1 का सही स्पष्टीकरण है।
(B*) कथन–1 सत्य है, कथन–2 सत्य है ; कथन–2, कथन–1 का सही स्पष्टीकरण नहीं है।
(C) कथन–1 सत्य है, कथन–2 असत्य है।
(D) कथन–1 असत्य है, कथन–2 सत्य है।
(E) सभी कथन असत्य है।

Sol. If we interchange any two rows of a determinant in set B, its value becomes –1, hence it becomes a member of set C.
$\Rightarrow$ Number of elements in set B is equal to number of elements in set C.
$\Rightarrow$ Statement - 1 is true.
Also $B \cap C = \phi \subset A$ $\Rightarrow$ statement-2 is true.

But statement-2 is not a correct explanation of statement-1.

Hindi यदि समुच्चय B में सारणिक की किन्हीं दो पंक्तियों को आपस में बदल दिया जाता हो, तो इसका मान –1 हो जाता है, अतः यह C का एक सदस्य बन जाता है।

$\Rightarrow$ समुच्चय B में अवयवों की संख्या समुच्चय C में अवयवों की संख्या के बराबर है।

$\Rightarrow$ वक्तव्य- 1 सत्य है।

अतः $B \cap C = \phi \subset A$ $\Rightarrow$ वक्तव्य- 2 सत्य है।

लेकिन वक्तव्य-2, वक्तव्य-1 का सही स्पष्टीकरण नहीं है

Section (C) : Cofactor matrix, adj matrix and inverse of matrix.
खण्ड (C) : सहखण्ड आव्यूह, सहखण्डज आव्यूह एवं आव्यूह का प्रतिलोम

C-1. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, then adj A =

यदि $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ हो, तो adj A =

(A*) $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$

Sol. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Matrix formed by cofactors of $A = C = \begin{bmatrix} 1 & -2 \\ -2 & +1 \end{bmatrix}$

Transpose of Matrix $C = C^T = \begin{bmatrix} 1 & -2 \\ -2 & +1 \end{bmatrix}$

Adjoint of matrix $A = C^T = \text{adj } A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$

Hindi. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

A के सहखण्ड से बना आव्यूह $A = C = \begin{bmatrix} 1 & -2 \\ -2 & +1 \end{bmatrix}$

C का परिवर्त आव्यूह $C = C^T = \begin{bmatrix} 1 & -2 \\ -2 & +1 \end{bmatrix}$

A का सहखण्डज आव्यूह $= C^T = \text{adj } A = \begin{bmatrix} 1 & -2 \\ -2 & +1 \end{bmatrix}$

C-2. Identify statements S_1, S_2, S_3 in order for true(T)/false(F)

S_1 : If $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then adj A = A'

S_2 : If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

S_3 : If B is a non-singular matrix and A is a square matrix, then det $(B^{-1} AB)$ = det (A)

S_1 : यदि $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ हो, तो adj A = A'

S_2 : यदि $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ हो, तो $A^{-1} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

S_3 : यदि B एक व्युत्क्रमणीय मैट्रिक्स और A एक वर्ग मैट्रिक्स हो, तो $\det(B^{-1} AB) = \det(A)$

क्रम में बताइये कि S_1, S_2, S_3 सत्य(T)/असत्य(F) है।

(A) TTF (B) FTT (C*) TFT (D) TTT

Sol. **S_1 :** $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Matrix formed by Cofactors of A= $C = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore$ $\text{Adj } A = C^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = A'$

Hindi. **S_1 :** $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A के सहखण्ड से बना आव्यूह $C = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\therefore$ $\text{Adj } A = C^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = A'$

S_2 : $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$; $|A| = abc$

$\text{adj}(A) = \begin{bmatrix} bc & 0 & 0 \\ 0 & ac & 0 \\ 0 & 0 & ab \end{bmatrix}$; $A^{-1} = \dfrac{\text{adj}(A)}{(A)} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$

S_3 : $\det(B^{-1} AB)$ Since (चूँकि) $\det(AB) = \det A . \det B$

$= \det B^{-1} . \det AB = \det B^{-1} . \det A . \det B = \dfrac{1}{\det B} \det A \det B = \det A$

C-3. If A, B are two $n \times n$ non-singular matrices, then

(A*) AB is non-singular (B) AB is singular

(C) $(AB)^{-1} = A^{-1} B^{-1}$ (D) $(AB)^{-1}$ does not exist

यदि $n \times n$ क्रम के दो व्युत्क्रमणीय आव्यूह A एवं B हो, तो–

(A) AB व्युत्क्रमणीय हैं। (B) AB अव्युत्क्रमणीय हैं। (C) $(AB)^{-1} = A^{-1} B^{-1}$ (D) $(AB)^{-1}$ विद्यमान नहीं हैं।

Sol. $|A| \neq 0$ and एवं $|B| \neq 0$ $\Rightarrow$ $|AB| \neq 0$

$\therefore$ AB is non singular. $\therefore$ AB व्युत्क्रमणीय है।

C-4. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and X be a matrix such that A = BX, then X is equal to

(A*) $\frac{1}{2}\begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}$ (B) $\frac{1}{2}\begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}$ (C) $\begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}$ (D) none of these

माना $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$ एवं $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ एवं मैट्रिक्स X इस प्रकार है कि A = BX हो, तो X =

(A*) $\frac{1}{2}\begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}$ (B) $\frac{1}{2}\begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}$ (C)) $\begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}$ (D) इनमें से कोई नहीं

Sol. $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$; $A = BX \Rightarrow B^{-1}A = B^{-1}(BX) \Rightarrow B^{-1}A = X$

$X = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$; $X = \frac{1}{2}\begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}$

C-5. Let $\mathbf{A} = \begin{bmatrix} -1 & 2 & -3 \\ -2 & 0 & 3 \\ 3 & -3 & 1 \end{bmatrix}$ be a matrix, then (det **A**) x (adj $\mathbf{A}^{-1}$) is equal to

(A) $O_{3\times 3}$ (B) I_3 (C*) $\begin{bmatrix} -1 & 2 & -3 \\ -2 & 0 & 3 \\ 3 & -3 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 3 & -3 & 1 \\ 3 & 0 & -2 \\ -1 & 2 & -3 \end{bmatrix}$

माना $\mathbf{A} = \begin{bmatrix} -1 & 2 & -3 \\ -2 & 0 & 3 \\ 3 & -3 & 1 \end{bmatrix}$ का एक आव्यूह हो, तो (det **A**) x (adj $\mathbf{A}^{-1}$) का मान है–

(A) $O_{3\times 3}$ (B) I_3 (C*) $\begin{bmatrix} -1 & 2 & -3 \\ -2 & 0 & 3 \\ 3 & -3 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 3 & -3 & 1 \\ 3 & 0 & -2 \\ -1 & 2 & -3 \end{bmatrix}$

Sol. $|A| \cdot (\text{adj}(A^{-1})) = |A|\,(|A^{-1}| \cdot (A^{-1})^{-1}) = |A A^{-1}| \cdot A = A.$

C-6. If D is a determinant of order three and Δ is a determinant formed by the cofactors of determinant D ; then following statement is **True/False**

(i) $\Delta = D^2$ (ii) D = 0 implies $\Delta = 0$
(iii) if D = 27, then Δ is perfect cube (iv) if D = 27, then Δ is perfect square

यदि D एक तीन क्रम का सारणिक हो तथा Δ, D के सहखण्डों से निर्मित सारणिक हो, तो निम्न कथन है, सत्य/असत्य

(i) $\Delta = D^2$
(ii) D = 0, तो $\Delta = 0$
(iii) यदि D = 27 हो, तो Δ पूर्ण घन है।
(iv) यदि D = 27 हो तो Δ एक पूर्णवर्ग है।

(A) FTTT (B) TFTT (C*) TTTT (D) TTFT

Sol. For n^{th} order determinant $\Delta = |C_{ij}| = D^{n-1}$
(A) For 3^{rd} order determinant $\Delta = D^{3-1} = D^2$ ... (1)
(B) From (1) if D = 0 then $\Delta = 0$
(C) $\Delta = 27 = 3^3$
$\Delta = (3^3)^2 = 3^6$ (a perfect cube)

Hindi. n^{th} क्रम की सारणिक के लिए $\Delta = |C_{ij}| = D^{n-1}$
(A) 3^{rd} क्रम की सारणिक के लिए $\Delta = D^{3-1} = D^2$ ... (1)
(B) (1) से यदि D = 0 हो तो $\Delta = 0$
(C) $\Delta = 27 = 3^3$
$\Delta = (3^3)^2 = 3^6$ (एक पूर्णघन)

Section (D) : Characteristic equation and system of equations
खण्ड (D) : अभिलाक्षणिक समीकरण एवं समीकरणों का निकाय

D-1. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ is a root of polynomial $x^3 - 6x^2 + 7x + k = 0$, then the value of k is

यदि $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ बहुपद $x^3 - 6x^2 + 7x + k = 0$ का एक मूल है तब k का मान है–

(A*) 2 (B) 4 (C) –2 (D) 1

Sol. $|A - \lambda I| = 0$

$$\begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{bmatrix} = 0 \quad \Rightarrow \quad (1 - \lambda)\,[(2 - \lambda)(3 - \lambda)] + 2[-2(2 - \lambda)] = 0$$

$(1 - \lambda)\,[\lambda^2 - 5\lambda + 6] + 4\,(\lambda - 2) = 0 \quad \Rightarrow \quad \lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0$

$x^3 - 6x^2 + 7x + 2 = 0 \quad \Rightarrow \quad k = 2$

D-2 If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (where $bc \neq 0$) satisfies the equations $x^2 + k = 0$, then

(A*) $a + d = 0$ & $k = |A|$ (B) $a - d = 0$ & $k = |A|$

(C) $a + d = 0$ & $k = -|A|$ (D) $a + d \neq 0$ & $k = |A|$

यदि $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (जहाँ $bc \neq 0$) समीकरण $x^2 + k = 0$ को संतुष्ट करता हो, तो –

(A*) $a + d = 0$ & $k = |A|$ (B) $a - d = 0$ & $k = |A|$

(C) $a + d = 0$ & $k = -|A|$ (D) $a + d \neq 0$ & $k = |A|$

Sol. $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ $bc \neq 0$

Characteristic equation is $|A - xI| = 0$ (अभिलाक्षणिक समीकरण है $|A - xI| = 0$)

$$\begin{vmatrix} a-x & b \\ c & d-x \end{vmatrix} = 0 \qquad (a - x)\,(d - x) - bc = 0$$

$x^2 - x\,(a + d) + ad - bc = 0$

On comparing with the given equation $x^2 + k = 0$ (दी गई समीकरण से तुलना करने पर $x^2 + k = 0$)

$a + d = 0,\ k = ad - bc = |A|$

D-3. If the system of equations $x + 2y + 3z = 4$, $x + py + 2z = 3$, $x + 4y + \mu z = 3$ has an infinite number of solutions and solution triplet is

(A) $p = 2,\ \mu = 3$ and $(5 - 4\lambda,\ \lambda - 1,\ \lambda)$

(B) $p = 2,\ \mu = 4$ and $(5 - 4\lambda,\ \frac{\lambda - 1}{2},\ 2\lambda)$

(C) $3p = 2\mu$ and $(5 - 4\lambda,\ \lambda - 1,\ 2\lambda)$

(D*) $p = 4,\ \mu = 2$ and $(5 - 4\lambda,\ \frac{\lambda - 1}{2},\ \lambda)$

यदि समीकरण निकाय $x + 2y + 3z = 4$, $x + py + 2z = 3$, $x + 4y + \mu z = 3$ के अनन्त हल है तथा हलत्रिक् हो, तो–

(A) $p = 2,\ \mu = 3$ तथा $(5 - 4\lambda,\ \lambda - 1,\ \lambda)$ (B) $p = 2,\ \mu = 4$ तथा $(5 - 4\lambda,\ \frac{\lambda - 1}{2},\ 2\lambda)$

(C) $3p = 2\mu$ तथा $(5 - 4\lambda,\ \lambda - 1,\ 2\lambda)$ (D*) $p = 4,\ \mu = 2$ तथा $(5 - 4\lambda,\ \frac{\lambda - 1}{2},\ \lambda)$

Sol. $$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 1 & p & 2 \\ 1 & 4 & \mu \end{vmatrix} \quad \Delta_x = \begin{vmatrix} 4 & 2 & 3 \\ 3 & p & 2 \\ 3 & 4 & \mu \end{vmatrix} \quad ; \quad \Delta_y = \begin{vmatrix} 1 & 4 & 3 \\ 1 & 3 & 2 \\ 1 & 3 & \mu \end{vmatrix} \quad \Delta_z = \begin{vmatrix} 1 & 2 & 4 \\ 1 & p & 3 \\ 1 & 4 & 3 \end{vmatrix}$$

For infinite no. of solution $\Delta = \Delta_x = \Delta_y = \Delta_z = 0 \quad \Rightarrow \mu = 2,\ p = 4$

अनन्त हलों के लिए $\Delta = \Delta_x = \Delta_y = \Delta_z = 0 \quad \Rightarrow \mu = 2,\ p = 4$

D-4. Let λ and α be real. Find the set of all values of λ for which the system of linear equations have infinite solution $\forall$ real values of α.
माना λ और α वास्तविक है। λ के सभी मानों का समुच्चय ज्ञात कीजिये जिसके लिये रैखिक समीकरणों का निकाय अनन्त हल रखता है, सभी वास्तविक α के लिए,

$\lambda x + (\sin\alpha)y + (\cos\alpha) z = 0$
$x + (\cos\alpha)y + (\sin\alpha) z = 0$
$-x + (\sin\alpha) y + (\cos\alpha) z = 0$

(A) $(-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$ (B*) -1
(C) $(-5, -\sqrt{2})$ (D) None of these इनमें से कोई नहीं

Sol. For non-trivial solution अशून्य हल के लिए

$$\begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda = \sin 2\alpha + \cos 2\alpha$$

$$-\sqrt{2} \le \sin 2\alpha + \cos 2\alpha \le \sqrt{2} \quad \Rightarrow \quad -\sqrt{2} \le \lambda \le \sqrt{2}\ .$$

D-5. The value of '2k' for which the set of equations $3x + ky - 2z = 0$, $x + ky + 3z = 0$, $2x + 3y - 4z = 0$ has a non – trivial solution over the set of rational is:
(A) 31 (B) 32 (C*) 33 (D) 34

2k का वह मान जिसके लिये समीकरण निकाय $3x + ky - 2z = 0$, $x + ky + 3z = 0$, $2x + 3y - 4z = 0$ का अशून्य हल (परिमेय संख्याओं के समुच्चय में से) विद्यमान है, होगा।
(A) 31 (B) 32 (C*) 33 (D) 34

Sol. $3x + ky - 2z = 0$; $x + ky + 3z = 0$; $2x + 3y - 4z = 0$

$$\Delta = \begin{vmatrix} 3 & k & -2 \\ 1 & k & 3 \\ 2 & 3 & -4 \end{vmatrix} = 0 \quad \therefore \quad k = \frac{33}{2}$$

PART - III : MATCH THE COLUMN

भाग - III : कॉलम को सुमेलित कीजिए (MATCH THE COLUMN)

1.✍ **Column I** **Column II**

(A) $\begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0$, then x = (p) 2

(B) If A is a square matrix of order 3 × 3 and k is a scalar, then adj (kA) = k^m adj A, then m is (q) – 2

(C) If $A = \begin{bmatrix} 2 & \mu \\ \mu^2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} \gamma & 7 \\ 49 & \delta \end{bmatrix}$ here (A – B) is upper triangular matrix then number of possible values of μ are (r) 1

(D)✍ If $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = k\, abc\, (a+b+c)^3$ then the value of k is (s) $-\frac{9}{8}$

स्तम्भ I स्तम्भ II

(A) $\begin{bmatrix}1 & x & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\3 & 2 & 5\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = 0$ हो, तो x = (p) 2

(B) यदि A एक 3 × 3 क्रम की वर्ग आव्यूह और k अदिश हो, तो adj (kA) = k^m adj A, तब m होगा (q) – 2

(C) यदि $A = \begin{bmatrix}2 & \mu\\ \mu^2 & 3\end{bmatrix}$ तथा $B = \begin{bmatrix}\gamma & 7\\ 49 & \delta\end{bmatrix}$ यहाँ (A – B) ऊपरी त्रिभुजाकार आव्यूह है तब μ के संभावित मानों की संख्या है (r) 1

(D) यदि $\begin{vmatrix}(b+c)^2 & a^2 & a^2\\ b^2 & (c+a)^2 & b^2\\ c^2 & c^2 & (a+b)^2\end{vmatrix} = k\,abc\,(a+b+c)^3$ हो, तो k का मान है– (s) $-\dfrac{9}{8}$

Ans. (A) → (s), (B) → (p), (C) → (p), (D) → (p)

Sol. (A) $\begin{bmatrix}1 & x & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\3 & 2 & 5\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = 0 \Rightarrow [1 + 4x + 3\ \ 2 + 5x + 2\ \ 3 + 6x + 5]\begin{bmatrix}1\\2\\3\end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix}4+4x & 5x+4 & 6x+8\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = 0 \Rightarrow 4 + 4x + 10x + 8 + 18x + 24 = 0$

$\Rightarrow 32x + 36 = 0 \Rightarrow x = -9/8$

(B) kA adj (kA) = |kA| I_n
kA adj (kA) = k^n |A| I_n
kA adj (kA) = k^n A adj A
Pre-multiplying A^{-1}
A^{-1} से पूर्व गुणन करने पर
adj (kA) = k^{n-1} adj A

(C) $(A - \lambda I) = 0 \Rightarrow \lambda^2 - 4\lambda + 5 = 0$
$\therefore A^2 - 4A + 5I = 0$

(D) Let a = b = c = 1 $\Rightarrow \begin{vmatrix}4 & 1 & 1\\1 & 4 & 1\\1 & 1 & 4\end{vmatrix} = k.27$

54 = 27 k $\Rightarrow$ k = 2

Alter : $\begin{vmatrix}(b+c)^2 & a^2 & a^2\\ b^2 & (c+a)^2 & b^2\\ c^2 & c^2 & (a+b)^2\end{vmatrix}$

Applying $C_1 \to C_2 - C_3$, $C_2 \to C_2 - C_3$

$(a+b+c)^2\begin{vmatrix}b+c-a & 0 & a^2\\ 0 & c+a-b & b^2\\ c-b-a & c-a-b & (a+b)^2\end{vmatrix}$

Applying $R_3 \to R_3 - (R_1 + R_2)$

$(a+b+c)^2\begin{vmatrix}b+c-a & 0 & a^2\\ 0 & c+a-b & b^2\\ -2b & -2a & 2ab\end{vmatrix} = \dfrac{(a+b+c)^2}{ab}\begin{vmatrix}a(b+c-a) & 0 & a^2\\ 0 & b(c+a-b) & b^2\\ -2ab & -2ab & 2ab\end{vmatrix}$

$$2ab\,(a+b+c)^2 \begin{vmatrix} (b+c-a) & 0 & a \\ 0 & (c+a-b) & b \\ -1 & -1 & 1 \end{vmatrix} = 2abc\,(a+b+c)^3$$

Hindi (A) $\begin{bmatrix} 1 & x & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0$ $\Rightarrow$ $[1 + 4x + 3\ \ 2 + 5x + 2\ \ 3 + 6x + 5]\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0$

$\Rightarrow$ $\begin{bmatrix} 4+4x & 5x+4 & 6x+8 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0$ $\Rightarrow$ $4 + 4x + 10x + 8 + 18x + 24 = 0$

$\Rightarrow$ $32x + 36 = 0$ $\Rightarrow$ $x = -9/8$

(B) $kA\ \text{adj}\ (kA) = |kA|\ I_n$

$kA\ \text{adj}\ (kA) = k^n\ |A|\ I_n$

$kA\ \text{adj}\ (kA) = k^n\ A\ \text{adj}\ A$

A^{-1} से पूर्व गुणन करने पर

$\text{adj}\ (kA) = k^{n-1}\ \text{adj}\ A$

(C) $(A - \lambda I) = 0$ $\Rightarrow$ $\lambda^2 - 4\lambda + 5 = 0$

$\therefore$ $A^2 - 4A + 5I = 0$

(D) माना $a = b = c = 1$ $\Rightarrow$ $\begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix} = k.27$

$54 = 27\,k$ $\Rightarrow$ $k = 2$

वैकल्पिक : $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$

संक्रिया $C_1 \to C_2 - C_3$, $C_2 \to C_2 - C_3$ से

$$(a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-b-a & c-a-b & (a+b)^2 \end{vmatrix}$$

संक्रिया $R_3 \to R_3 - (R_1 + R_2)$ से

$$(a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix} = \frac{(a+b+c)^2}{ab}\begin{vmatrix} a(b+c-a) & 0 & a^2 \\ 0 & b(c+a-b) & b^2 \\ -2ab & -2ab & 2ab \end{vmatrix}$$

$$2ab\,(a+b+c)^2 \begin{vmatrix} (b+c-a) & 0 & a \\ 0 & (c+a-b) & b \\ -1 & -1 & 1 \end{vmatrix} = 2abc\,(a+b+c)^3$$

2. **Column – I** **Column – II**

(A) If A and B are square matrices of order 3 × 3, where $|A| = 2$ and $|B| = 1$, then $|(A^{-1})\ .\ \text{adj}\ (B^{-1})\ .\ \text{adj}\ (2A^{-1})| =$ (p) 7

(B) If A is a square matrix such that $A^2 = A$ and $(I + A)^3 = I + kA$, then k is equal to (q) 8

(C) Matrix $\begin{bmatrix} a & b & (a\alpha - b) \\ b & c & (b\alpha - c) \\ 2 & 1 & 0 \end{bmatrix}$ is non invertible ($b^2 \neq ac$) if -2α is (r) 0

(D) If $A = [a_{ij}]_{3\times3}$ is a scalar matrix with $a_{11} = a_{22} = a_{33}$ (s) – 1

= 2 and A(adjA) = kI_3 then k is

स्तम्भ I		स्तम्भ II	
(A)	यदि A तथा B , 3 × 3 क्रम के वर्ग आव्यूह है, जहाँ |A| = 2 तथा |B| = 1 है, तो $\|(A^{-1}) . adj (B^{-1}) . adj (2A^{-1})\|$ =	(p)	7
(B)	यदि A एक वर्ग आव्यूह इस प्रकार है कि $A^2 = A$ एवं $(I + A)^3 = I + kA$, तो k =	(q)	8
(C)	आव्यूह $\begin{bmatrix} a & b & (a\alpha - b) \\ b & c & (b\alpha - c) \\ 2 & 1 & 0 \end{bmatrix}$ अप्रतिलोमीय है। ($b^2 \neq ac$) यदि -2α है	(r)	0
(D)	यदि $A = [a_{ij}]_{3\times3}$ एक अदिश आव्यूह है, जहाँ $a_{11} = a_{22} = a_{33} = 2$ तथा A(adjA) = kI_3, तब k =	(s)	– 1

Ans. (A) → (q), (B) → (p), (C) → (s), (D) → (q)

Sol. (A) $|(A^{-1})\ adj\ (B^{-1})\ .\ adj(2A^{-1})| = |A^{-1}|\ .\ |adj\ B^{-1}|\ .\ |adj\ .\ 2A^{-1}| = \frac{1}{|A|}.|B^{-1}|^2|2A^{-1}|^2 = \frac{2^6}{8} = 8$

(B) $I^3 + 3I^2A + 3IA^2 + A^3 = I + 3A + 3A + A = I + 7A \Rightarrow k = 7$

(C) Taking $C_3 \to C_3 - (C_1\alpha - C_2)$

we get

$$|A| = \begin{vmatrix} a & b & 0 \\ b & c & 0 \\ 2 & 1 & -2\alpha+1 \end{vmatrix} = (1 - 2\alpha)\ (ac - b^2)$$

∴ non-invertible if $\alpha = \frac{1}{2}$ or if a, b, c are in G.P.

(D) A Adj A = $|A|\ I_3 = kI_3 \Rightarrow k = |A| = 8$

Hindi. (A) $|(A^{-1})\ adj\ (B^{-1})\ .\ adj(2A^{-1})| = |A^{-1}|\ .\ |adj\ B^{-1}|\ .\ |adj\ .\ 2A^{-1}| = \frac{1}{|A|}.|B^{-1}|^2|2A^{-1}|^2 = \frac{2^6}{8} = 8$

(B) $I^3 + 3I^2A + 3IA^2 + A^3 = I + 3A + 3A + A = I + 7A \Rightarrow k = 7$

(C) संक्रिया $C_3 \to C_3 - (C_1\alpha - C_2)$ से

$$|A| = \begin{vmatrix} a & b & 0 \\ b & c & 0 \\ 2 & 1 & -2\alpha+1 \end{vmatrix} = (1 - 2\alpha)\ (ac - b^2)$$

∴ अप्रतिलोमीय होगा यदि $\alpha = \frac{1}{2}$ या यदि a, b, c गुणोत्तर श्रेणी में हो।

(D) A Adj A = $|A|\ I_3 = kI_3 \Rightarrow k = |A| = 8$

Exercise-2

✍ **Marked questions are recommended for Revision.**
✍ **चिन्हित प्रश्न दोहराने योग्य प्रश्न है।**

PART - I : ONLY ONE OPTION CORRECT TYPE
भाग-I : केवल एक सही विकल्प प्रकार (ONLY ONE OPTION CORRECT TYPE)

1. Two matrices A and B have in total 6 different elements (none repeated) . How many different matrices A and B are possible such that product AB is defined.
A और B दो आव्यूह में कुल 6 भिन्न-भिन्न अवयव है (पुनरावृत्ति नहीं) कितने भिन्न-भिन्न आव्यूह A और B संभव है जबकि गुणन AB परिभाषित है।
(A) 5(6!) (B) 3(6!) (C) 12(6!) (D*) 8 (6!)

Sol. Let (माना) $A = [a_{ij}]\ p \times q$
$B = [b_{ij}]\ q \times r$
$p \times q + q \times r = 6$
$q\ (p + r) = 6$

if (यदि) $q = 1 \Rightarrow p + r = 6 \begin{cases} 1 & 5 \\ 2 & 4 \\ 3 & 3 \\ 4 & 3 \\ 5 & 1 \end{cases}$

if यदि $q = 2 \Rightarrow p + r = 3 \begin{cases} 2 & 1 \\ 1 & 2 \end{cases}$

if यदि $q = 3 \Rightarrow p + r = 2\ \{1,1$
$(5 + 2 + 1) \times 6! = 8 \times 6\,!$

2.✍ If $X = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then value of X^n is, (where n is natural number)

(A) $\begin{bmatrix} 3n & -4n \\ n & -n \end{bmatrix}$ (B) $\begin{bmatrix} 2+n & 5-n \\ n & -n \end{bmatrix}$ (C) $\begin{bmatrix} 3^n & (-4)^n \\ 1^n & (-1)^n \end{bmatrix}$ (D*) $\begin{bmatrix} 2n+1 & -4n \\ n & -(2n-1) \end{bmatrix}$

यदि $X = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ हो, तो X^n का मान है (जहाँ n प्राकृत संख्या है)–

(A) $\begin{bmatrix} 3n & -4n \\ n & -n \end{bmatrix}$ (B) $\begin{bmatrix} 2+n & 5-n \\ n & -n \end{bmatrix}$ (C) $\begin{bmatrix} 3^n & (-4)^n \\ 1^n & (-1)^n \end{bmatrix}$ (D*) $\begin{bmatrix} 2n+1 & -4n \\ n & -(2n-1) \end{bmatrix}$

Sol. $X = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \Rightarrow X^2 = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$
For n = 2 option A, B, C are not satisified. Hence option D is correct.
n = 2 के लिए A, B, C संतुष्ट नही है अतः D सही है।

3. If A and B are two matrices such that AB = B and BA = A, then $A^2 + B^2 =$
यदि A और B दो मैट्रिक्स इस प्रकार है कि AB = B और BA = A हो, तो $A^2 + B^2 =$
(A) 2AB (B) 2BA (C*) A + B (D) AB

Sol. AB = B
Premultiply both sides by B
$BAB = B^2 \Rightarrow AB = B^2 \Rightarrow B = B^2$
Similarly
$BA = A \Rightarrow ABA = A^2 \Rightarrow BA = A^2$
$\Rightarrow A = A^2$

Hindi. AB = B
B के पूर्व गुणन दोनो तरफ करने पर
$BAB = B^2 \Rightarrow AB = B^2 \Rightarrow B = B^2$
इसी प्रकार
$BA = A \Rightarrow ABA = A^2 \Rightarrow BA = A^2 \Rightarrow A = A^2$

4. Find number of all possible ordered sets of two (n × n) matrices **A** and **B** for which **AB** – **BA** = I

(A) infinite (B) n^2 (C) n! (D*) zero

(n × n) क्रम के आव्यूह **A** तथा **B** के क्रमित युग्मों के कितने समुच्चय संभव है जबकि **AB** – **BA** = I

(A) अनन्त (B) n^2 (C) n! (D*) शून्य

Ans. No such ordered pair is possible. (ऐसा कोई क्रमित युग्म संभव नहीं है)

Sol. Assume **C** = **AB – BA**
If **C** = I
then trace (**C**) = 1 + 1 + ... + 1 = n
But trace (**C**) = 0 (trace (**AB**) = trace (.**BA**))
Which is a contradiction
Hence no such ordered pair is possible.

Hindi. माना **C** = **AB – BA**
यदि **C** = I
तो अनुरेख (**C**) = 1 + 1 + ... + 1 = n
लेकिन अनुरेख (**C**) = 0 (अनुरेख (**AB**) = अनुरेख (.**BA**))
जो कि विरोधाभाष है
अतः ऐसा कोई क्रमित युग्म संभव नहीं है

5. If B, C are square matrices of order n and if A = B + C, BC = CB, $C^2 = O$, then which of following is true for any positive integer N.
यदि B, C, n क्रम के वर्ग आव्यूह है यदि A = B + C, BC = CB, $C^2 = O$, तब किसी धनात्मक पूर्णांक N के लिए निम्न में से कौनसा सही है

(A*) $A^{N+1} = B^N (B + (N + 1) C)$ (B) $A^N = B^N (B + (N + 1) C)$
(C) $A^{N+1} = B (B + (N + 1) C)$ (D) $A^{N+1} = B^N (B + (N + 2) C)$

Sol. $A^{N+1} = (B + C)^{N+1}$
We can expand $(B + C)^{N+1}$ like binomial expansion as BC = CB.
$(B + C)^{N+1}$ को द्विपद के अनुसार विस्तार करने पर BC = CB.
$\therefore \quad (B + C)^{N+1} = {}^{N+1}C_0 B^{N+1} + {}^{N+1}C_1 B^N C + {}^{N+1}C_2 B^{N-1} C^2 + ... + C^{N+1}.$
$= B^{N+1} + (N + 1) B^N C + 0 + 0 .... + 0 = B^N (B + (N +1) C).$

6. How many 3 × 3 skew symmetric matrices can be formed using numbers –2, –1, 1, 2, 3, 4, 0 (any number can be used any number of times but 0 can be used at most 3 times)
संख्याओं –2, –1, 1, 2, 3, 4, 0 की सहायता से 3 × 3 क्रम के विषम सममित आव्यूह को बनायी जाती है (किसी संख्या को कितनी भी बार उपयोम में ली जा सकती है परन्तु 0 को अधिक से अधिक 3 बार उपयोम में लिया जा सकता है।

(A) 8 (B) 27 (C*) 64 (D) 54

Sol. 4 × 4 × 4 = 64

7. If A is a skew - symmetric matrix and n is an even positive integer, then A^n is
(A*) a symmetric matrix (B) a skew-symmetric matrix
(C) a diagonal matrix (D) none of these

यदि A एक विषम सममित आव्यूह और n एक धनात्मक सम पूर्णांक हो, तो A^n है –
(A) एक सममित आव्यूह (B) एक विषम सममित आव्यूह
(C) एक विकर्ण आव्यूह (D) इनमें से कोई नहीं

Sol. $A^T = -A \Rightarrow (A^n)^T = (AAA ------- A)^T = (A^T A^T A^T ------- A^T) = (A^T)^n$ for all $n \in N$

$(-A)^n = (-1)^n A^n \Rightarrow (A^n)^T = \begin{cases} A^n & \text{if n is even} \\ -A^n & \text{if n is odd} \end{cases}$

Hindi. $A^T = -A$ $\Rightarrow$ $(A^n)^T = (AAA \text{-------} A)^T = (A^T A^T A^T \text{-------} A^T) = (A^T)^n$, सभी के लिए $n \in N$

$(-A)^n = (-1)^n A^n$ $\Rightarrow$ $(A^n)^T = \begin{cases} A^n, \text{ यदि } n \text{ सम है} \\ -A^n, \text{ यदि } n \text{ विषम है} \end{cases}$

8. Number of 3 × 3 non symmetric matrix A such that $A^T = A^2 - I$ and $|A| \neq 0$, equals to

(A*) 0 (B) 2 (C) 4 (D) Infinite

3 × 3 क्रम का आव्यूह A जो कि सममित नहीं है, की संख्या होगी जबकि $A^T = A^2 - I$ और $|A| \neq 0$, बराबर है–

(A*) 0 (B) 2 (C) 4 (D) अनन्त

Sol. $A = (A^T)^2 - I = (A^2 - I)^2 - I$

$\Rightarrow A^4 - 2A^2 - A = O \Rightarrow A^3 - 2A - I = O$

$\Rightarrow (A + I)(A^2 - A - I) = O \Rightarrow (A + I)(A^T - A) = O$

Because $|A + I| \neq 0 \Rightarrow A^T = A \Rightarrow$ A is symmetric which contradict given condition

$\Rightarrow$ no such matrix possible

Hindi. $A = (A^T)^2 - I = (A^2 - I)^2 - I$

$\Rightarrow A^4 - 2A^2 - A = O \Rightarrow A^3 - 2A - I = O$

$\Rightarrow (A + I)(A^2 - A - I) = O \Rightarrow (A + I)(A^T - A) = O$

क्योंकि $|A + I| \neq 0 \Rightarrow A^T = A \Rightarrow A$ सममित है जो दिए गए प्रतिबन्ध का विरोधाभास है।

इस प्रकार का कोई आव्यूह संभव है।

9. Matrix **A** is such that $\mathbf{A}^2 = 2\mathbf{A} - I$, where I is the identity matrix. Then for $n \geq 2$, $\mathbf{A}^n$ =

(A*) $n\mathbf{A} - (n-1)I$ (B) $n\mathbf{A} - I$ (C) $2^{n-1}\mathbf{A} - (n-1)I$ (D) $2^{n-1}\mathbf{A} - I$

आव्यूह **A** इस प्रकार है कि $\mathbf{A}^2 = 2\mathbf{A} - I$, जहाँ I तत्समक आव्यूह है तब $n \geq 2$ के लिए $\mathbf{A}^n$ =

(A*) $n\mathbf{A} - (n-1)I$ (B) $n\mathbf{A} - I$ (C) $2^{n-1}\mathbf{A} - (n-1)I$ (D) $2^{n-1}\mathbf{A} - I$

Sol. $\mathbf{A}^2 - 2\mathbf{A} + I = 0$ $\Rightarrow$ $(A - I)^2 = 0$

$\mathbf{A}^n = (\mathbf{A} - I + I)^n = {}^n\mathbf{C}_0 (\mathbf{A} - I)^n + \ldots\ldots + {}^n\mathbf{C}_{n-2} (\mathbf{A} - I)^2 \cdot I^{n-2} + {}^n\mathbf{C}_{n-1}(\mathbf{A} - I) \cdot I^{n-1} + {}^n\mathbf{C}_n I^n$

$= 0 + 0 + \ldots\ldots\ldots + 0 + n(\mathbf{A} - I) + I = n\mathbf{A} - (n-1)I$

10. If $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$ and $x = P^TQ^{2005}P$, then x is equal to

यदि $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ और $Q = PAP^T$ तथा $x = P^TQ^{2005}P$ हो, तो x =

(A*) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 4+2005\sqrt{3} & 6015 \\ 2005 & 4-2005\sqrt{3} \end{bmatrix}$

(C) $\frac{1}{4}\begin{bmatrix} 2+\sqrt{3} & 1 \\ -1 & 2-\sqrt{3} \end{bmatrix}$ (D) $\frac{1}{4}\begin{bmatrix} 2005 & 2-\sqrt{3} \\ 2+\sqrt{3} & 2005 \end{bmatrix}$

Sol. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PA\,P^T$ and $X = P^TQ^{2005}\,P$

We observe that $Q = PA\,P^T$

$Q^2 = (P\,A\,P^T)(P\,A\,P^T) = PA\,(P^TP)\,A\,P^T = PA\,(IA)\,P^T = P\,A^2\,P^T$

Proceeding in the same way, we get

$Q^{2005} = P\,A^{2005}\,P^T$

Also $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

And proceeding in the same way

$A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

$$P^TP = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Now,

$X = P^T Q^{2005} P = P^T (PA^{2005} P^T) P = (P^T P) A^{2005} (P^TP) = I A^{2005} I$

$$= A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$$

Hindi $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ एवं $Q = PA P^T$ तथा $X = P^T Q^{2005} P$

$Q = PA P^T$ $\quad Q^2 = (P A P^T) (P A P^T) = PA (P^TP) A P^T = PA (IA) P^T = P A^2 P^T$

इसी तरह

$Q^{2005} = P A^{2005} P^T$

तथा $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

तथा इसी प्रकार

$A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

$$P^TP = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$X = P^T Q^{2005} P = P^T (PA^{2005} P^T) P = (P^T P) A^{2005} (P^TP) = I A^{2005} I = A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

11. Let $\Delta = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{vmatrix}$, then

(A) Δ is independent of θ (B*) Δ is indepedent of ϕ
(C) Δ is a constant (D) none of these

माना $\Delta = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{vmatrix}$, तो

(A) Δ , θ से स्वतन्त्र है। (B) Δ , ϕ से स्वतन्त्र है।
(C) Δ एक अचर है। (D) इनमें से कोई नहीं

Sol. $\Delta = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{vmatrix}$; $\Delta = \sin^2\theta\cos\theta \begin{vmatrix} \cos\phi & \sin\phi & \cot\theta \\ \cos\phi & \sin\phi & -\tan\theta \\ -\sin\phi & \cos\phi & 0 \end{vmatrix}$

Applying $R_1 \to R_1 - R_2$ (संक्रिया $R_1 \to R_1 - R_2$ से)

$\Delta = \sin^2\theta\cos\theta \begin{vmatrix} 0 & 0 & \cot\theta + \tan\theta \\ \cos\phi & \sin\phi & -\tan\theta \\ -\sin\phi & \cos\phi & 0 \end{vmatrix}$; $\Delta = \sin\theta$

12. $\Delta = \begin{vmatrix} 1+a^2+a^4 & 1+ab+a^2b^2 & 1+ac+a^2c^2 \\ 1+ab+a^2b^2 & 1+b^2+b^4 & 1+bc+b^2c^2 \\ 1+ac+a^2c^2 & 1+bc+b^2c^2 & 1+c^2+c^4 \end{vmatrix}$ is equal to

$\Delta = \begin{vmatrix} 1+a^2+a^4 & 1+ab+a^2b^2 & 1+ac+a^2c^2 \\ 1+ab+a^2b^2 & 1+b^2+b^4 & 1+bc+b^2c^2 \\ 1+ac+a^2c^2 & 1+bc+b^2c^2 & 1+c^2+c^4 \end{vmatrix}$ का मान है –

(A*) $(a-b)^2 (b-c)^2 (c-a)^2$ (B) $2(a-b)(b-c)(c-a)$
(C) $4(a-b)(b-c)(c-a)$ (D) $(a+b+c)^3$

Sol. $\Delta = \begin{vmatrix} 1+a^2+a^4 & a+ab+a^2b^2 & 1+ac+a^2c^2 \\ 1+ab+a^2b^2 & 1+b^2+b^4 & 1+bc+b^2c^2 \\ 1+ac+a^2c^2 & 1+bc+b^2c^2 & 1+c^2+c^4 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$= (a-b)^2 (b-c)^2 (c-a)^2$

13. If $D = \begin{vmatrix} a^2+1 & ab & ac \\ ba & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$ then D =

(A*) $1 + a^2 + b^2 + c^2$ (B) $a^2 + b^2 + c^2$ (C) $(a+b+c)^2$ (D) none

यदि $D = \begin{vmatrix} a^2+1 & ab & ac \\ ba & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$ तो D =

(A) $1 + a^2 + b^2 + c^2$ (B) $a^2 + b^2 + c^2$ (C) $(a+b+c)^2$ (D) इनमें से कोई नहीं

Sol. $\begin{bmatrix} a^2+1 & ab & ac \\ ba & b^2+1 & bc \\ ca & cb & c^2+1 \end{bmatrix} = \frac{1}{abc}\begin{bmatrix} a(a^2+1) & a^2b & a^2c \\ b^2a & b(b^2+1) & b^2c \\ c^2a & c^2b & c(c^2+1) \end{bmatrix} = \frac{abc}{abc}\begin{bmatrix} a^2+1 & a^2 & a^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{bmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$ (संक्रिया $R_1 \to R_1 + R_2 + R_3$ से)

$(a^2+b^2+c^2+1)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$ & $C_3 \to C_3 - C_1$ (संक्रिया $C_2 \to C_2 - C_1$ & $C_3 \to C_3 - C_1$ से)

$(a^2+b^2+c^2+1)\begin{vmatrix} 1 & 0 & 0 \\ b^2 & 1 & 0 \\ c^2 & 0 & 1 \end{vmatrix} = (a^2 + b^2 + c^2 + 1)$

14. Value of the $\Delta = \begin{vmatrix} a^3-x & a^4-x & a^5-x \\ a^5-x & a^6-x & a^7-x \\ a^7-x & a^8-x & a^9-x \end{vmatrix}$ is

$\Delta = \begin{vmatrix} a^3-x & a^4-x & a^5-x \\ a^5-x & a^6-x & a^7-x \\ a^7-x & a^8-x & a^9-x \end{vmatrix}$ का मान है –

(A*) 0 (B) $(a^3-1)(a^6-1)(a^9-1)$
(C) $(a^3+1)(a^6+1)(a^9+1)$ (D) $a^{15}-1$

Sol. $\begin{vmatrix} a^3-x & a^4-x & a^5-x \\ a^5-x & a^6-x & a^7-x \\ a^7-x & a^8-x & a^9-x \end{vmatrix}$

Applying $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$ (संक्रिया $R_3 \to R_2$, $R_2 \to R_2 - R_1$ से)

$$\begin{vmatrix} a^3-x & a^4-x & a^5-x \\ a^5-a^3 & a^6-a^4 & a^7-a^5 \\ a^7-a^5 & a^8-a^6 & a^9-a^7 \end{vmatrix} = \begin{vmatrix} a^3-x & a^4-x & a^5-x \\ a^2-1 & a^3-a & a^4-a^2 \\ a^2-1 & a^3-a & a^4-a^2 \end{vmatrix} = a^3a^5 = 0$$

15. If $\Delta_1 = \begin{vmatrix} 2a & b & e \\ 2d & e & f \\ 4x & 2y & 2z \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} f & 2d & e \\ 2z & 4x & 2y \\ e & 2a & b \end{vmatrix}$, then the value of $\Delta_1 - \Delta_2$ is

यदि $\Delta_1 = \begin{vmatrix} 2a & b & e \\ 2d & e & f \\ 4x & 2y & 2z \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} f & 2d & e \\ 2z & 4x & 2y \\ e & 2a & b \end{vmatrix}$, हो, तो $\Delta_1 - \Delta_2$ का मान है –

(A) $x + \frac{y}{2} + z$ (B) 2 (C*) 0 (D) 3

Sol. $\Delta_1 = \begin{vmatrix} 2a & b & e \\ 2d & e & f \\ 4x & 2y & 2z \end{vmatrix} = 4\begin{vmatrix} a & b & e \\ d & e & f \\ x & y & z \end{vmatrix}$; $\Delta_2 = \begin{vmatrix} f & 2d & e \\ 2z & 4x & 2y \\ e & 2a & b \end{vmatrix} = 4\begin{vmatrix} f & d & e \\ z & x & y \\ e & a & b \end{vmatrix}$

$C_1 \leftrightarrow C_1$ followed by तथा $C_2 \leftrightarrow C_3$ से

$\Delta_2 = \begin{vmatrix} d & e & f \\ x & y & z \\ a & b & c \end{vmatrix}$

$R_1 \leftrightarrow R_2$ followed by तथा $R_1 \leftrightarrow R_3$ $= \begin{vmatrix} a & b & e \\ d & e & f \\ x & y & z \end{vmatrix}$ से

$\Delta_2 = \Delta_1 \quad \Rightarrow \quad \Delta_1 - \Delta_2 = 0$

16. From the matrix equation AB = AC, we conclude B = C provided:

(A) A is singular (B*) A is non–singular (C) A is symmetric (D) A is a square

आव्यूह समीकरण AB = AC से यह निष्कर्ष निकलता है कि B = C जबकि–

(A) A अव्युत्क्रमणीय हो। (B) A व्युत्क्रमणीय हो। (C) A सममित हो। (D) A वर्ग मैट्रिक्स हो।

Sol. AB = AC

Pre-multiplying $A^{-1} \Rightarrow B = C$ hence A must be invertible matrix.

Hindi. AB = AC

A^{-1} से पूर्वगुणन करने पर $\Rightarrow B = C$ अतः यह प्रतिलोमिय आव्यूह होना चाहिए।

17. Let $A = \begin{bmatrix} -2 & 7 & \sqrt{3} \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}$ and $A^4 = \lambda.I$, then λ is

माना $A = \begin{bmatrix} -2 & 7 & \sqrt{3} \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}$ तथा $A^4 = \lambda.I$, तब λ का मान है –

(A) – 16 (B*) 16 (C) 8 (D) –8

Sol. $|A - x\,I| = 0$

$$\begin{bmatrix} -2-x & 7 & \sqrt{3} \\ 0 & -x & -2 \\ 0 & 2 & -x \end{bmatrix} = 0 \Rightarrow (-2-x)\,(x^2+4) = 0$$

$(x + 2)\,(x^2 + 4) = 0 \Rightarrow x^3 + 2x^2 + 4x + 8 = 0$

According to calley hamiltan theorem

कैले हेमिल्टन प्रमेय से

$A^3 + 2A^2 + 4A + 8I = 0$ .....(1)

$A^4 + 2A^3 + 4A^2 + 8A = 0$ .....(2)

$\Rightarrow A^4 + 2(-8I) = 0 \Rightarrow A^4 = 16\,I$

18. If A is 3 × 3 square matrix whose characterstic polynomial equations is $\lambda^3 - 3\lambda^2 + 4 = 0$ then trace of adjA is

(A*) 0 (B) 3 (C) 4 (D) – 3

यदि A, 3 × 3 क्रम के वर्ग आव्यूह है जिसकी अभिलाक्षणिक बहुपद समीकरण $\lambda^3 - 3\lambda^2 + 4 = 0$ है तब adjA का अनुरेख है–

(A*) 0 (B) 3 (C) 4 (D) – 3

Sol. $A^3 - 3A^2 + 4I = O$ ....(i)

Multiply equation of $(A^{-1})^3$ we get $I - 3A^{-1} + 4(A^{-1})^3 = O$

$\Rightarrow 16I - 12(-4A^{-1}) + (-4A^{-1})^3 = O$

$\Rightarrow (\text{adj}A)^3 - 12(\text{adj }A) - 16I = O$

$\Rightarrow$ Trace of adjA is zero

Hindi: $A^3 - 3A^2 + 4I = O$ ....(i)

$(A^{-1})^3$ से गुणा करने पर we get $I - 3A^{-1} + 4(A^{-1})^3 = O$

$\Rightarrow 16I - 12(-4A^{-1}) + (-4A^{-1})^3 = O$

$\Rightarrow (\text{adj}A)^3 - 12(\text{adj }A) - 16I = O$

$\Rightarrow$ Trace of adjA is zero

19. If a, b, c are non zeros, then the system of equations

$$(\alpha + a)x + \alpha y + \alpha z = 0$$
$$\alpha x + (\alpha + b)y + \alpha z = 0$$
$$\alpha x + \alpha y + (\alpha + c)z = 0$$

has a non-trivial solution if

(A*) $\alpha^{-1} = -(a^{-1} + b^{-1} + c^{-1})$ (B) $\alpha^{-1} = a + b + c$

(C) $\alpha + a + b + c = 1$ (D) none of these

यदि a, b, c अशून्य हैं, तो समीकरण निकाय

$$(\alpha + a)x + \alpha y + \alpha z = 0$$
$$\alpha x + (\alpha + b)y + \alpha z = 0$$
$$\alpha x + \alpha y + (\alpha + c)z = 0$$

के अशून्य हल होगें यदि

(A) $\alpha^{-1} = -(a^{-1} + b^{-1} + c^{-1})$ (B) $\alpha^{-1} = a + b + c$

(C) $\alpha + a + b + c = 1$ (D) इनमें से कोई नहीं

Sol. For non-trivial solution (अशून्य हल के लिए)

$$\begin{vmatrix} (\alpha+a) & \alpha & \alpha \\ \alpha & \alpha+b & \alpha \\ \alpha & \alpha & \alpha+c \end{vmatrix} = 0$$

Taking α as common from each row (प्रत्येक पंक्ति से α उभयनिष्ठ लेते है)

$$\Rightarrow \alpha^3 \begin{vmatrix} 1+\frac{a}{\alpha} & 1 & 1 \\ 1 & 1+\frac{b}{\alpha} & 1 \\ 1 & 1 & 1+\frac{c}{\alpha} \end{vmatrix} = 0$$

Applying $C_1 \to C_1 - C_3$, $C_2 \to C_2 - C_3$ and expanding

(संक्रिया $C_1 \to C_1 - C_3$, $C_2 \to C_2 - C_3$ तथा प्रसार करते है)

$$\Rightarrow \quad \alpha^3\left[\frac{ab}{\alpha^2} + \frac{bc}{\alpha^2} + \frac{ac}{\alpha^2} + \frac{abc}{\alpha^3}\right] = 0 \quad \Rightarrow \quad \frac{1}{\alpha} = -\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

20. **STATEMENT-1 :** If A = $\begin{bmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+xc & b^2+x^2 & bc-ax \\ ac-bx & bc+ax & c^2+x^2 \end{bmatrix}$ and B = $\begin{bmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{bmatrix}$, then |A| =|B|^2.

STATEMENT-2 : If A^c is cofactor matrix of a square matrix A of order n then $|A^c| = |A|^{n-1}$.

(A*) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1

(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1

(C) STATEMENT-1 is true, STATEMENT-2 is false

(D) STATEMENT-1 is false, STATEMENT-2 is true

(E) Both STATEMENTS are false

कथन 1 : यदि A = $\begin{bmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+xc & b^2+x^2 & bc-ax \\ ac-bx & bc+ax & c^2+x^2 \end{bmatrix}$ और B = $\begin{bmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{bmatrix}$ हो, तो |A| =|B|^2.

कथन 2 : यदि n क्रम के वर्ग आव्यूह A का सहखण्ड आव्यूह A^c हो, तो $|A^c| = |A|^{n-1}$.

(A*) कथन–1 सत्य है, कथन–2 सत्य है ; कथन–2, कथन–1 का सही स्पष्टीकरण है।

(B) कथन–1 सत्य है, कथन–2 सत्य है ; कथन–2, कथन–1 का सही स्पष्टीकरण नहीं है।

(C) कथन–1 सत्य है, कथन–2 असत्य है।

(D) कथन–1 असत्य है, कथन–2 सत्य है।

(E) सभी कथन असत्य है।

Sol. B = $\begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}$

Transpose matrix formed by cofactors of B = $\begin{bmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+cx & b^2+x^2 & bc-ax \\ ac-bx & bc+ax & c^2+x^2 \end{bmatrix}$

B के सहखण्डों से बने आव्यूह का परिवर्त आव्यूह B = $\begin{bmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+cx & b^2+x^2 & bc-ax \\ ac-bx & bc+ax & c^2+x^2 \end{bmatrix}$

$\Rightarrow$ adj B = A $\Rightarrow$ $|\text{adj}B| = |A| \Rightarrow |B|^2 = |A|$

21. Let A = $\begin{bmatrix} a & 0 & b \\ 1 & e & 1 \\ c & 0 & d \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ where a,b, c, d, e $\in$ {0, 1}

then number of such matrix A for which system of equation AX = O have unique solution.

(A) 16 (B*) 6 (C) 5 (D) none

माना A $\begin{bmatrix} a & 0 & b \\ 1 & e & 1 \\ c & 0 & d \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ जहाँ a,b, c, d, e $\in$ {0, 1}

तो इस प्रकार के आव्यूहों A की संख्या जिसके लिये समीकरण निकाय AX = O अद्वितीय हल रखता हो–

(A) 16 (B*) 6 (C) 5 (D) इनमें से कोई नहीं

Sol. $|A| \neq 0$ $\Rightarrow$ a(ed – 0) + b (0 – ce)

$|A|$ = aed – bce = e (ad – bc) $\neq$ 0

e = 1 and तथा ad – bc $\neq$ 0

ad – bc = 1 if यदि [ad = 1, bc = 0] Total = 3

ad – bc = –1 if यदि [ad = 0, bc = 1] Total = 3

Total योग = 3 + 3 = 6

22. If the system of equations ax + y + z = 0,. x + by + z = 0 and x + y + cz = 0, where a, b, c ≠ 1, has a non–trivial solution, then the value of $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}$ is

(A*) 1 (B) 2 (C) 3 (D) 4

यदि समीकरण निकाय ax + y + z = 0,. x + by + z = 0 और x + y + cz = 0 के लिए जहाँ a, b, c ≠ 1, सार्थक (Non-trivial) हल हो, तो $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}$ का मान है।

(A*) 1 (B) 2 (C) 3 (D) 4

Ans. 1

Sol. $\Delta = \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$

$R_1 \to R_1 - R_2$ & $R_2 \to R_2 - R_3$

$\Delta = \begin{vmatrix} a-1 & 1-b & 0 \\ 0 & b-1 & 1-c \\ 1 & 1 & c \end{vmatrix} = 0 \quad \Rightarrow \quad (a-1)[(b-1)c-(1-c)+(1-b)(1-c)] = 0$

$\Rightarrow \frac{c}{1-c}+\frac{1}{1-b}+\frac{1}{1-a} = 0 \quad \Rightarrow \quad \frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c} = 1$

PART-II: NUMERICAL VALUE QUESTIONS

भाग-II : संख्यात्मक प्रश्न (NUMERICAL VALUE QUESTIONS)

INSTRUCTION :

- The answer to each question is **NUMERICAL VALUE** with two digit integer and decimal upto two digit.
- If the numerical value has more than two decimal places **truncate/round-off** the value to **TWO** decimal placed.

निर्देश :

- इस खण्ड में प्रत्येक प्रश्न का उत्तर **संख्यात्मक मान** के रूप में है जिसमें दो पूर्णांक अंक तथा दो अंक दशमलव के बाद में है।
- यदि संख्यात्मक मान में दो से अधिक दशमलव स्थान है, तो संख्यात्मक मान को दशमलव के **दो** स्थानों तक **ट्रंकेट/राउंड ऑफ (truncate/round-off)** करें।

1. Let X be the solution set of the equation

$A^x = I$, where $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ and I is the unit matrix and $X \subset N$ then the minimum value of $\sum_x (\cos^x\theta + \sin^x\theta)$, $\theta \in R$ is :

माना X समीकरण $A^x = I$ का हल समुच्चय है जहाँ $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ तथा I इकाई आव्यूह है तथा $X \subset N$ तब , $\sum_x (\cos^x\theta + \sin^x\theta)$, $\theta \in R$ का न्यूनतम मान है :

Ans. **02.00**

Sol. $A^2 = I \quad \Rightarrow \quad A^4 = A^6 = \text{.........} = I$

Now अब $A^x = I \Rightarrow \quad x = 2, 4, 6, 8, ...$

$\sum_x (\cos^x\theta + \sin^x\theta) = \cos^2\theta + \cos^4\theta + ... + \sin^2\theta + \sin^4\theta + \sin^6\theta + ...$

$= \frac{\cos^2\theta}{1-\cos^2\theta} + \frac{\sin^2\theta}{1-\sin^2\theta} = \cot^2\theta + \tan^2\theta$

which has minimum value 2.

जिसका न्यूनतम मान 2 है।

2. If A is a diagonal matrix of order 3 × 3 is commutative with every square matrix of order 3 × 3 under multiplication and tr(A) = 10, then the value of |A| is :

यदि A, एक 3 × 3 क्रम का विकर्ण आव्यूह है जो गुणन के अर्न्तगत 3 × 3 क्रम के प्रत्येक वर्ग आव्यूह के साथ क्रमविनिमेय है तथा tr(A) = 10 तब |A| का मान है :

Ans. **37.03 or 37.04**

Sol. A diagonal matrix is commutatives with every square matrix if it is scalar matrix. So every diagonal element is $\frac{10}{3}$.

so $\quad |A| = \frac{1000}{27}$

Hindi. एक विकर्ण आव्यूह, प्रत्येक वर्ग आव्यूह के साथ क्रम विनिमेय है यदि यह अदिश आव्यूह है इसलिए प्रत्येक विकर्ण अवयव $\frac{10}{3}$ है। इसलिए $|A| = \frac{1000}{27}$

3. A, is a (3×3) diagonal matrix having integral entries such that det(A) = 120, number of such matrices is 10n. Then n is :

A, एक (3×3) क्रम का विकर्ण आव्यूह है जिसमें पूर्णांक अवयव इस प्रकार है कि det(**A**) = 120 तथा इस प्रकार के आव्यूहों की संख्या 10n है तब n का मान है :

Ans. **36.00**

Sol. given (दिया है) $\quad \text{I} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \Rightarrow abc = 120 = 2^3 \times 3^1 \times 5^1$

Case - I All are Positive (सभी धनात्मक है) $= {}^5C_2 \times {}^3C_2 \times {}^3C_2 = 90$

Case - I $\quad$ one Positive and two negative (एक धनात्मक तथा दो ऋणात्मक है)

$= 3 \times ({}^5C_2 \times {}^3C_2 \times {}^3C_2) = 270$

So number of possible matrices = 90 + 270 = 360

अतः संम्भव आव्यूहों की संख्या = 90 + 270 = 360

4. If $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \geq 0$, where a, b, c ∈ R^+, then $\frac{c}{a+b}$ is

यदि $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \geq 0$, जहाँ a, b, c ∈ R^+, तब $\frac{c}{a+b}$ है–

Ans. **00.50**

Sol. Let $\quad$ माना $\quad \Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \quad$ Operate : $C_1 \rightarrow C_1 + C_2 + C_3$ से

$\Rightarrow \Delta = \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+a) & a+b & b+c \\ 2(a+b+c) & b+c & c+a \end{vmatrix} = 2(a+b+c) \begin{vmatrix} 1 & c+a & a+b \\ 1 & a+b & b+c \\ 1 & b+c & c+a \end{vmatrix}$

Operate : $R_2 \rightarrow R_2 - R_1$; $R_3 \rightarrow R_3 - R_1$ से

$$\Rightarrow D = 2(a+b+c)\begin{vmatrix} 1 & c+a & a+b \\ 0 & b-c & c-a \\ 0 & b-a & c-b \end{vmatrix}$$

$= 2(a+b+c)\ 1.\begin{vmatrix} b-c & c-a \\ b-a & c-b \end{vmatrix}$ open w.r.t. R_1 R_1 के लिए प्रसार

$= 2(a+b+c)\,[(b-c)(c-b)-(c-a)(b-a)] = 2(a+b+c)\,[bc-b^2-c^2+cb-(cb-ac-ab+a^2)]$

$= 2(a+b+c)\,(ab+bc+ca-a^2-b^2-c^2) \le 0 \quad \Rightarrow \quad a=b=c$

5. If a_1, a_2, a_3 , 5, 4, a_6, a_7, a_8, a_9 are in H.P. and $\mathbf{D} = \begin{vmatrix} a_1 & a_2 & a_3 \\ 5 & 4 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$, then the value of **D** is

Ans. **02.38**

यदि a_1, a_2, a_3 , 5, 4, a_6, a_7, a_8, a_9 हरात्मक श्रेणी मे हैं तथा $\mathbf{D} = \begin{vmatrix} a_1 & a_2 & a_3 \\ 5 & 4 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$, तो 21**D** का मान है

Sol. $\mathbf{D} = \begin{vmatrix} a_1 & a_2 & a_3 \\ 5 & 4 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} \quad \Rightarrow \quad \frac{1}{a_1}, \frac{1}{a_2}, \ldots\ldots, \frac{1}{5}, \frac{1}{4}, \ldots\ldots, \frac{1}{a_9}$ are in A.P.

$d = \frac{1}{4} - \frac{1}{5} = \frac{1}{20} \quad \Rightarrow \quad \frac{1}{5} = \frac{1}{a} + \frac{3}{20} \quad \Rightarrow \quad \frac{1}{a_1} = \frac{1}{20} \quad \therefore \quad \frac{1}{a_n} = \frac{1}{a_1} + \frac{(n-1)}{20} = \frac{n}{20}$

$\Rightarrow \ a_n = \frac{20}{n}$

Hence (अतः), $\mathbf{D} = \begin{vmatrix} 20 & \frac{20}{2} & \frac{20}{3} \\ \frac{20}{4} & \frac{20}{5} & \frac{20}{6} \\ \frac{20}{7} & \frac{20}{8} & \frac{20}{9} \end{vmatrix} = \frac{(20)^3}{4\times 7}\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 1 & \frac{4}{5} & \frac{2}{3} \\ 1 & \frac{7}{8} & \frac{7}{9} \end{vmatrix}$

$R_1 \rightarrow R_1 - R_2$ and (तथा) $R_2 \rightarrow R_2 - R_3$ से

$$= \frac{(20)^3}{4\times 7}\begin{vmatrix} 0 & \frac{-3}{10} & \frac{-1}{3} \\ 0 & \frac{-3}{40} & \frac{-1}{9} \\ 1 & \frac{7}{8} & \frac{7}{9} \end{vmatrix} = \frac{50}{21} \Rightarrow \qquad \mathbf{21D} = 50$$

6. If यदि $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = k(\alpha a + \beta b + \gamma c)^3$, then तब $(2\alpha + \beta + \gamma)^k$ is $(\alpha, \beta, \gamma, k \in z^+)$ है

Ans. **16.00**

Sol. LHS = $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

Operate संक्रिया : $C_1 \rightarrow C_1 + C_2 + C_3$ लगाने पर

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix} = 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Operate संक्रिया : $R_2 \rightarrow R_2 - R_1$; $R_3 \rightarrow R_3 - R_1$ लगाने पर

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & (a+b+c) & 0 \\ 0 & 0 & (c+a+b) \end{vmatrix}$$

Expand by C_1 से प्रसार करने पर

$$= 2(a+b+c).1. \begin{vmatrix} a+b+c & 0 \\ 0 & a+b+c \end{vmatrix} = 2(a+b+c)^3 \Rightarrow \qquad k = 2 \text{ and और } \alpha = \beta = \gamma = 1$$

7. If A is a square matrix of order 3 and A′ denotes transpose of matrix A, $A'A = I$ and det A = 1, then det $(A - I)$ must be equal to

Ans. **00.00**

यदि A एक तीन क्रम का वर्ग आव्यूह है तथा A′ आव्यूह A की परिवर्त आव्यूह को प्रदर्शित करता है तथा $A'A = I$ एवं det A = 1 तब det $(A - I)$ बराबर होना चाहिए–

Sol. $A'A = I$

$|A - I| = |A - A'A| \quad \Rightarrow \quad |A - I| = |A|\,|I - A'|$

$\Rightarrow \quad |A - I| = -1\,.\,|A' - I| \quad \Rightarrow \quad |A - I| = -|A - I| \quad \Rightarrow \quad |A - I| = 0$

8. Suppose A is a matrix such that $A^2 = A$ and $(I + A)^6 = I + kA$, then k is

माना A एक आव्यूह इस प्रकार है कि $A^2 = A$ तथा $(I + A)^6 = I + kA$ हो, तो k का मान है–

Ans. **63.00**

Sol. $A^2 = A \quad \Rightarrow \quad A^{-1} A^2 = A^{-1}A \quad \Rightarrow \quad A = I$

$\therefore \quad (I + A)^6 = (I + I)^6 = (2I)^6 = 64\,I = I + KI = (K + 1)\,I \qquad \therefore\ K + 1 = 64 \quad \Rightarrow \quad K = 63$

9. If $\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = 64$, then $(ab + bc + ac)$ is :

यदि $\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = 64$, तब $(ab + bc + ac)$ है :

Ans. **04.00**

Sol. $$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} \Rightarrow \frac{1}{abc}\begin{vmatrix} -abc & ab^2+abc & ac^2+abc \\ a^2b+abc & -abc & bc^2+abc \\ a^2c+abc & b^2c+abc & -abc \end{vmatrix}$$

$$\Rightarrow \frac{abc}{abc}\begin{vmatrix} -bc & ab+ac & ac+ab \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + R_2 + R_3$ (संक्रिया $R_1 \rightarrow R_1 + R_2 + R_3$ से)

$$(ab+bc+ca)\begin{vmatrix} 1 & 1 & 1 \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - C_1$, $C_3 \rightarrow C_3 - C_1$ (संक्रिया $C_2 \rightarrow C_2 - C_1$, $C_3 \rightarrow C_3 - C_1$ से)

$$\Rightarrow (ab+bc+ca)\begin{vmatrix} 1 & 0 & 0 \\ ab+bc & -(ab+bc+ac) & 0 \\ ac+bc & 0 & -(ab+bc+ca) \end{vmatrix} = (ab+bc+ca)^3$$

10. Let $f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$ then value of $f(\pi/12) + f(\pi/6) + f(\pi/4) + f(\pi/3) + f(\pi/2)$ is

यदि $f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$ हो तो $f(\pi/12) + f(\pi/6) + f(\pi/4) + f(\pi/3) + f(\pi/2)$ का मान है –

Ans. **22.92 or 22.93**

Sol. $f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$ Applying $C_1 \to C_1 + C_2 + C_3$ (संक्रिया $C_1 \to C_1 + C_2 + C_3$ से)

$= (2 + 4 \sin 2x)$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$= (2 + 4 \sin 2x) \begin{vmatrix} 1 & \cos^2 x & 4\sin 2x \\ 1 & 1+\cos^2 x & 4\sin 2x \\ 1 & \cos^2 x & 1+4\sin 2x \end{vmatrix}$

$f(x) = (2 + 4\sin 2x) \begin{vmatrix} 1 & \cos^2 x & 4\sin 2x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \Rightarrow \quad f(x)_{max} = 6$

11. If $U_n = \begin{vmatrix} n & 1 & 5 \\ n^2 & 2N+1 & 2N+1 \\ n^3 & 3N^2 & 3N+1 \end{vmatrix}$ and $\sum_{n=1}^{N} U_n = \lambda \sum_{n=1}^{N} n^2$, then λ is

यदि $U_n = \begin{vmatrix} n & 1 & 5 \\ n^2 & 2N+1 & 2N+1 \\ n^3 & 3N^2 & 3N+1 \end{vmatrix}$ और $\sum_{n=1}^{N} U_n = \lambda \sum_{n=1}^{N} n^2$, तब λ है–

Ans. **02.00**

Sol. $U_n = \begin{vmatrix} n & 1 & 5 \\ n^2 & 2N+1 & 2N+1 \\ n^3 & 3N^2 & 3N+1 \end{vmatrix} \Rightarrow \sum_{n=1}^{N} U_n = \begin{vmatrix} \frac{N(N+1)}{2} & 1 & 5 \\ \frac{N(N+1)(2N+1)}{6} & (2N+1) & (2N+1) \\ \left[\frac{N(N+1)}{2}\right]^2 & 3N^2 & (3N+1) \end{vmatrix}$

$= \frac{N(N+1)(2N+1)}{2} \begin{vmatrix} 1 & 1 & 5 \\ \frac{1}{3} & 1 & 1 \\ \frac{N(N+1)}{2} & 3N^2 & 3N+1 \end{vmatrix}$

Applying $R_1 \to R_1 - R_2$

$\frac{N(N+1)(2N+1)}{2} \begin{vmatrix} \frac{2}{3} & 0 & 4 \\ \frac{1}{3} & 1 & 1 \\ \frac{N(N+1)}{2} & 3N^2 & 3N+1 \end{vmatrix} = 2 \sum_{n=1}^{N} n^2$

12. The absolute value of a for which system of equations, $a^3x + (a + 1)^3y + (a + 2)^3z = 0$, $ax + (a + 1) y + (a + 2) z = 0$, $x + y + z = 0$, has a non–zero solution is:

Hindi. a का निरपेक्ष मान होगा जिसके लिए समीकरणों $a^3x + (a + 1)^3y + (a + 2)^3z = 0$,
$ax + (a + 1) y + (a + 2) z = 0$, $x + y + z = 0$, का निकाय, अशून्य हल रखता है –

Ans. **01.00**

Sol. $\begin{vmatrix} a^3 & (a+1)^3 & (a+2)^3 \\ a & (a+1) & (a+2) \\ 1 & 1 & 1 \end{vmatrix} = 0$ for non - zero solution (अशून्य हल के लिए)

$C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

$\begin{vmatrix} a^3 & (a+1)^3 - a^3 & (a+2)^3 - a^3 \\ a & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 0$

$\Rightarrow 2((a + 1)^3 - a^3) - ((a + 2)^3 - a^3) = 0 \quad \Rightarrow 2 (3a^2 + 3a + 1) = 6a^2 + 12a + 8$

$\Rightarrow 6a + 2 = 12 a + 8 \quad \Rightarrow - 6 = 6a \quad \Rightarrow a = -1 \quad \Rightarrow \quad |a| = 1$

13. Consider the system of linear equations in x, y, z:

$(\sin 3\theta)\, x - y + z = 0$
$(\cos 2\theta)\, x + 4y + 3z = 0$
$2x + 7y + 7z = 0$

sum of all possible values of $\theta \in (0, \pi)$ for which this system has non – trivial solution, is

Hindi. माना कि x, y, z में रेखिक समीकरणों का निकाय है

$(\sin 3\theta)\, x - y + z = 0$
$(\cos 2\theta)\, x + 4y + 3z = 0$
$2x + 7y + 7z = 0$

$\theta \in (0, \pi)$ में समीकरण के सभी संभावितक मानों का योग होगा, जिसके लिए यह निकाय अशून्य हल रखता है।

Ans. **03.14**

Sol. For non-trivial solution अशून्य हल के लिए

$\Delta = 0$

$\begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$

$7\sin 3\theta + (7\cos 2\theta - 6) + (7\cos 2\theta - 8) = 0$

$\sin 3\theta + 2\cos 2\theta - 2 = 0$

$3\sin\theta - 4\sin^3\theta + 2 - 4\sin^2\theta - 2 = 0$

Let माना $\sin\theta = t$

$3t - 4t^3 - 4t^2 = 0$
$t(4t^2 + 4t - 3) = 0$
$t(4t^2 + 6t - 2t - 3) = 0$
$t(2t - 1)(2t + 3) = 0$

$t = 0 \qquad t = \frac{1}{2} \qquad \underbrace{t = \frac{-3}{2}}_{\text{not possible}} \qquad \underbrace{t = \frac{-3}{2}}_{\text{संभव नहीं}}$

$\sin\theta = 0 \quad \sin\theta = 1/2$

$\theta = n\pi \qquad \theta = n\pi + (-1)^n \frac{\pi}{6} \qquad n \in I \quad \Rightarrow \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

14. $A_1 = [\, a_1]$

$A_2 = \begin{bmatrix} a_2 & a_3 \\ a_4 & a_5 \end{bmatrix}$

$A_3 = \begin{bmatrix} a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} \\ a_{12} & a_{13} & a_{14} \end{bmatrix}$ ....$A_n = [........]$

Where $a_r = [\log_2 r]$ ([.] denotes greatest integer). Then trace of A_{10}

$$A_1 = [a_1]$$

$$A_2 = \begin{bmatrix} a_2 & a_3 \\ a_4 & a_5 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} \\ a_{12} & a_{13} & a_{14} \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots A_n = [\ldots\ldots]$$

जहाँ $a_r = [\log_2 r]$ ([.] महत्तम पूर्णांक को प्रदर्शित करता है) तो अनुरेख A_{10} का मान है :

Ans. **80.00**

Sol. First element in $\mathbf{A_{10}}$ is $a_{1^2+2^2+3^2+\ldots\ldots\ldots\ldots\ldots 9^2+1} = a_{286}$ and $[a_{286}] = 8$

and last element is a_{385}

$a_{385} = [\log_2 385] = 8$

Sum = 80

Hindi. $\mathbf{A_{10}}$ का प्रथम अवयव है $a_{1^2+2^2+3^2+\ldots\ldots\ldots\ldots\ldots 9^2+1} = a_{286}$ तथा $[a_{286}] = 8$

तथा अन्तिम अवयव a_{385} है

$a_{385} = [\log_2 385] = 8$ योगफल = 80

15. If $\left\{\frac{1}{2}(A - A' + I)\right\}^{-1} = \frac{2}{\lambda}\begin{bmatrix} \lambda-13 & -\frac{\lambda}{3} & \frac{\lambda}{3} \\ -17 & 10 & -1 \\ 7 & -11 & 5 \end{bmatrix}$ for $A = \begin{bmatrix} -2 & 3 & 4 \\ 5 & -4 & -3 \\ 7 & 2 & 9 \end{bmatrix}$, then λ is :

यदि $A = \begin{bmatrix} -2 & 3 & 4 \\ 5 & -4 & -3 \\ 7 & 2 & 9 \end{bmatrix}$ के लिए, $\left\{\frac{1}{2}(A - A' + I)\right\}^{-1} = \frac{2}{\lambda}\begin{bmatrix} \lambda-13 & -\frac{\lambda}{3} & \frac{\lambda}{3} \\ -17 & 10 & -1 \\ 7 & -11 & 5 \end{bmatrix}$, तब λ है–

Ans. **39.00**

Sol. $\left\{\frac{1}{2}(A - A' + I)\right\}^{-1}$ for के लिए $A = \begin{bmatrix} -2 & 3 & 4 \\ 5 & -4 & -3 \\ 7 & 2 & 9 \end{bmatrix}$

$$\frac{1}{2}(A - A^T + I)^{-1} = \left[\frac{1}{2}\begin{bmatrix} 1 & -2 & -3 \\ 2 & 1 & -5 \\ 3 & 5 & 1 \end{bmatrix}\right]^{-1} = \left(\frac{1}{2}B\right)^{-1} \Rightarrow \left|\frac{1}{2}B\right| = \frac{39}{8}$$

$$\text{Adj } B = \frac{1}{4}\begin{bmatrix} 26 & -17 & 7 \\ -13 & 10 & -11 \\ 13 & -1 & 5 \end{bmatrix}^T = \frac{1}{4}\begin{bmatrix} 26 & -13 & 13 \\ -17 & 10 & -1 \\ 7 & -11 & 5 \end{bmatrix} \Rightarrow \left|\frac{1}{2}B^{-1}\right| = \frac{2}{39}\begin{bmatrix} 26 & -13 & 13 \\ -17 & 10 & -1 \\ 7 & -11 & 5 \end{bmatrix}$$

16. Given $A = \begin{bmatrix} 2 & 0 & -\alpha \\ 5 & \alpha & 0 \\ 0 & \alpha & 3 \end{bmatrix}$ For $\alpha \in R - \{a, b\}$, A^{-1} exists and $A^{-1} = A^2 - 5bA + cI$, when $\alpha = 1$. The value of $a + b + c$ is :

दिया गया है $A = \begin{bmatrix} 2 & 0 & -\alpha \\ 5 & \alpha & 0 \\ 0 & \alpha & 3 \end{bmatrix}$, $\alpha \in R - \{a, b\}$ के लिए A^{-1} विद्यमान है तथा $A^{-1} = A^2 - 5bA + cI$, तब $\alpha = 1$ तब $a + b + c$ का मान है :

Ans. **12.20**

Sol. $A = \begin{vmatrix} 2 & 0 & -\alpha \\ 5 & \alpha & 0 \\ 0 & \alpha & 3 \end{vmatrix}$; $|A| = \begin{vmatrix} 2 & 0 & -\alpha \\ 5 & \alpha & 0 \\ 0 & \alpha & 3 \end{vmatrix} \neq 0$

$6\alpha - 5\alpha^2 \neq = 0 \quad \Rightarrow \quad \alpha(6 - 5\alpha) \neq 0$

$\alpha = 0, 6/5 \quad \therefore \quad \alpha \in R - \{0, 6/5\}$

For $\alpha = 1$ (के लिए)

$A = \begin{vmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix} \Rightarrow |A| = 6 - 5 = 1$; Adj $A = \begin{vmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{vmatrix}$

$\therefore \quad A^{-1} = \begin{vmatrix} 3 & -1 & 1 \\ -15 & 6 & 5 \\ 5 & -2 & 2 \end{vmatrix}$

By characteristic equation $|A - xI| = 0$ (अभिलाक्षणिक समीकरण से $|A - xI| = 0$)

$\begin{vmatrix} 2-x & 0 & -1 \\ 5 & 1-x & 0 \\ 0 & 1 & 3-x \end{vmatrix} = 0 \quad \Rightarrow x^3 - 6x^2 + 11x - 1 = 0$

By cayley hamilton theorem (हेमिल्टन प्रमेय से)

$A^3 - 6A^2 + 11A = I \Rightarrow A^{-1} = A^2 - 6A + 11.I$

17. Let a, b, c positive numbers. Find the number of solution of system of equations in x, y and z

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$; $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$;$-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ has finitely many solutions

माना a, b, c धनात्मक संख्याए है, x, y और z में समीकरणों

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$; $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$;$-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ का निकाय परिमित हल रखता है, के हलों की संख्या ज्ञात कीजिए।

Ans. **08.00**

Sol. $\Delta = \begin{vmatrix} \frac{1}{a^2} & \frac{1}{b^2} & \frac{-1}{c^2} \\ \frac{1}{a^2} & \frac{-1}{b^2} & \frac{1}{c^2} \\ \frac{-1}{a^2} & \frac{1}{b^2} & \frac{1}{c^2} \end{vmatrix}$ by $R_1 \to R_1 + R_2$, $R_2 \to R_2 + R_3$

$\Delta = \begin{vmatrix} \frac{2}{a^2} & 0 & 0 \\ 0 & 0 & \frac{2}{c^2} \\ \frac{-1}{a^2} & \frac{1}{b^2} & \frac{1}{c^2} \end{vmatrix} = \frac{2}{a^2}\left(\frac{-2}{c^2b^2}\right) = \frac{-4}{a^2b^2c^2}$

$\Delta \neq 0$ Here equations are of second degree so, more than one finite solution.

Hindi. $\Delta = \begin{vmatrix} \frac{1}{a^2} & \frac{1}{b^2} & \frac{-1}{c^2} \\ \frac{1}{a^2} & \frac{-1}{b^2} & \frac{1}{c^2} \\ \frac{-1}{a^2} & \frac{1}{b^2} & \frac{1}{c^2} \end{vmatrix}$ by $R_1 \to R_1 + R_2$, $R_2 \to R_2 + R_3$

$$\Delta = \begin{vmatrix} \frac{2}{a^2} & 0 & 0 \\ 0 & 0 & \frac{2}{c^2} \\ \frac{-1}{a^2} & \frac{1}{b^2} & \frac{1}{c^2} \end{vmatrix} = \frac{2}{a^2}\left(\frac{-2}{c^2b^2}\right) = \frac{-4}{a^2b^2c^2}$$

$\Delta \neq 0$ चूंकि यहाँ पर समीकरण द्वितीय घात की है अतः एक से ज्यादा परिमित हल होंगे।

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE
भाग - III : एक या एक से अधिक सही विकल्प प्रकार

1. Suppose a_1, a_2, a_3 are in A.P. and b_1, b_2, b_3 are in H.P. and let

$$\Delta = \begin{vmatrix} a_1-b_1 & a_1-b_2 & a_1-b_3 \\ a_2-b_1 & a_2-b_2 & a_2-b_3 \\ a_3-b_1 & a_3-b_2 & a_3-b_3 \end{vmatrix}, \text{ then}$$

(A*) Δ is independent of a_1, a_2, a_3,
(B*) $a_1 - \Delta, a_2 - 2\Delta, a_3 - 3\Delta$ are in A.P.
(C*) $b_1 + \Delta, b_2 + \Delta^2, b_3 + \Delta$ are in H.P.
(D*) Δ is independent of b_1, b_2, b_3

माना a_1, a_2, a_3 समान्तर श्रेढ़ी में हैं और b_1, b_2, b_3 हरात्मक श्रेढ़ी में हैं। यदि $\Delta = \begin{vmatrix} a_1-b_1 & a_1-b_2 & a_1-b_3 \\ a_2-b_1 & a_2-b_2 & a_2-b_3 \\ a_3-b_1 & a_3-b_2 & a_3-b_3 \end{vmatrix}$ हो, तो

(A) Δ , a_1, a_2, a_3, से स्वतन्त्र है।
(B) $a_1 - \Delta, a_2 - 2\Delta, a_3 - 3\Delta$ समान्तर श्रेढ़ी में है।
(C) $b_1 + \Delta, b_2 + \Delta^2, b_3 + \Delta$ हरात्मक श्रेढ़ी में हैं।
(D) Δ का मान b_1, b_2, b_3 से स्वतन्त्र है।

Sol. $\begin{vmatrix} a_1-b_1 & a_1-b_2 & a_1-b_3 \\ a_2-b_1 & a_2-b_2 & a_2-b_3 \\ a_3-b_1 & a_3-b_2 & a_3-b_3 \end{vmatrix}$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_2$

$$\begin{vmatrix} a_1-b_1 & a_1-b_2 & a_1 \\ a_2-a_1 & a_2-a_1 & a_2-a_1 \\ a_3-a_2 & a_3-a_2 & a_3-a_2 \end{vmatrix} = \begin{vmatrix} a_1-b_1 & a_1-b_2 & a_1 \\ d & d & d \\ d & d & d \end{vmatrix} = 0$$

$\Delta = 0$

2. Let $\theta = \frac{\pi}{5}$, $X = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, O is null maxtrix and I is an identity matrix of order 2 × 2, and if $I + X + X^2 + ... + X^n = O$, then n can be

माना $\theta = \frac{\pi}{5}$, $X = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, O शून्य आव्यूह है तथा I , 2 × 2 क्रम का तत्समक आव्यूह है यदि $I + X + X^2 + ... + X^n = O$, तब n का मान हो सकता है–

(A*) 9 (B*) 19 (C) 4 (D*) 29

Sol. $$X + X^2 + ... X^n = \begin{bmatrix} \cos\frac{\pi}{5} + \cos\frac{2\pi}{5} + ...+ \cos\frac{n\pi}{5} & -\left(\sin\frac{\pi}{5} + ...+ \sin\frac{n\pi}{5}\right) \\ \sin\frac{\pi}{5} + \sin\frac{2\pi}{5} + ...+ \cos\frac{n\pi}{5} & \cos\frac{\pi}{5} + ...+ \cos\frac{n\pi}{5} \end{bmatrix}$$

$$\sum_{r=1}^{n} \cos\frac{r\pi}{5} = -1 \text{ and और } \sum_{r=1}^{n} \sin\frac{r\pi}{5} = 0 \quad \Rightarrow \quad n = 9, 19, 29, ...$$

3. If $\Delta = \begin{vmatrix} x & 2y-z & -z \\ y & 2x-z & -z \\ y & 2y-z & 2x-2y-z \end{vmatrix}$, then

(A*) $x - y$ is a factor of Δ
(B*) $(x - y)^2$ is a factor of Δ
(C) $(x - y)^3$ is a factor of Δ
(D) Δ is independent of z

यदि $\Delta = \begin{vmatrix} x & 2y-z & -z \\ y & 2x-z & -z \\ y & 2y-z & 2x-2y-z \end{vmatrix}$ हो, तो

(A) $x - y$, Δ का एक गुणनखण्ड है। (B) $(x - y)^2$, Δ का एक गुणनखण्ड हैं।
(C) $(x - y)^3$, Δ का एक गुणनखण्ड है। (D) Δ , z से स्वतन्त्र है।

Sol. $\Delta = \begin{vmatrix} x & 2y-z & -z \\ y & 2x-z & -z \\ y & 2y-z & 2x-2y-z \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$ (संक्रिया $R_2 \to R_2 - R_1$ तथा $R_3 \to R_3 - R_1$ से)

$= \begin{vmatrix} x & 2y-z & -z \\ y-x & 2(x-y) & 0 \\ y-x & 0 & 2(x-y) \end{vmatrix} = (x-y)^2 \begin{vmatrix} x & 2y-z & -z \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix} = 4(x-y)^2 (x+y-z)$

So Δ is divisible by $(x - y)$ & $(x - y)^2$. (अतः Δ, $(x - y)$ तथा $(x - y)^2$ से विभाजित है)

4. Let a, b > 0 and $\Delta = \begin{vmatrix} -x & a & b \\ b & -x & a \\ a & b & -x \end{vmatrix}$, then

(A*) $a + b - x$ is a factor of Δ (B*) $x^2 + (a + b)x + a^2 + b^2 - ab$ is a factor of Δ
(C*) $\Delta = 0$ has three real roots if a = b (D) $a + b + x$ is a factor of Δ

माना a, b > 0 और $\Delta = \begin{vmatrix} -x & a & b \\ b & -x & a \\ a & b & -x \end{vmatrix}$ हो, तो

(A) $a + b - x$, Δ का एक गुणनखण्ड है। (B) $x^2 + (a + b)x + a^2 + b^2 - ab$, Δ का एक गुणनखण्ड है।
(C) यदि a = b हो, तो $\Delta = 0$ के तीन वास्तविक मूल है,
(D) $a + b + x$, Δ का एक गुणनखण्ड है।

Sol. $\Delta = \begin{vmatrix} -x & a & b \\ b & -x & a \\ a & b & -x \end{vmatrix} = (a+b-x) \begin{vmatrix} 1 & a & b \\ 1 & -x & a \\ 1 & b & -x \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ (संक्रिया $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ से)

$= (a+b-x) \begin{vmatrix} 1 & a & b \\ 0 & -(x+a) & a-b \\ 0 & b-a & -(x+b) \end{vmatrix} = (a+b-x)\{(x+a)(x+b) + (a-b)^2\}$

If (यदि) a = b then (है, तो) $(2a - x)\{(x + a)^2\} = 0$ then $x = -a$, $x = 2a$ i.e., अर्थात् x is real वास्तविक है.

5. The determinent $\Delta = \begin{vmatrix} b & c & b\alpha+c \\ c & d & c\alpha+d \\ b\alpha+c & c\alpha+d & a\alpha^3-c\alpha \end{vmatrix}$ is equal to zero if

(A) b, c, d are in A.P. (B*) b, c, d are in G.P.
(C) b, c, d are in H.P. (D*) α is a root of $ax^3 - bx^2 - 3cx - d = 0$

सारणिक $\begin{vmatrix} b & c & b\alpha+c \\ c & d & c\alpha+d \\ b\alpha+c & c\alpha+d & a\alpha^3-c\alpha \end{vmatrix}$ का मान शून्य होगा, यदि

(A) b, c, d स.श्रे. में हैं। (B) b, c, d गु.श्रे. में हैं।
(C) b, c, d ह.श्रे. में हैं। (D) α , $ax^3 - bx^2 - 3cx - d = 0$ का एक मूल है।

Sol. $\Delta = \begin{vmatrix} b & c & b\alpha+c \\ c & d & c\alpha+d \\ b\alpha+c & c\alpha+d & a\alpha^3-c\alpha \end{vmatrix} = 0$

Applying $C_3 \to C_3 - (C_1\alpha + C_2)$ (संक्रिया $C_3 \to C_3 - (C_1\alpha + C_2)$ से)

$$\Delta = \begin{vmatrix} b & c & 0 \\ c & d & 0 \\ b\alpha + c & c\alpha + d & a\alpha^3 - b\alpha^2 - 3c\alpha - d \end{vmatrix} = 0 \quad (a\alpha^3 - b\alpha^2 - 3c\alpha - d)\,(bd - c^2) = 0$$

$\therefore$ Either b, c, d in G.P. or α is root of $ax^3 - bx^2 - 3cx - d = 0$

($\therefore$ या तो b, c, d गुणोत्तर श्रेढी मे है या $ax^3 - bx^2 - 3cx - d = 0$ का मूल α है)

6. The determinant $\Delta = \begin{vmatrix} a^2(1+x) & ab & ac \\ ab & b^2(1+x) & bc \\ ac & bc & c^2(1+x) \end{vmatrix}$ is divisible by

सारणिक $\Delta = \begin{vmatrix} a^2(1+x) & ab & ac \\ ab & b^2(1+x) & bc \\ ac & bc & c^2(1+x) \end{vmatrix}$ का एक गुणनखण्ड है–

(A*) $x + 3$ (B) $(1 + x)^2$ (C*) x^2 (D) $x^2 + 1$

Sol. $\Delta = \begin{vmatrix} a^2(1+x) & ab & ac \\ ab & b^2(1+x) & bc \\ ac & bc & c^2(1+x) \end{vmatrix} = a^2b^2c^2 \begin{vmatrix} (1+x) & 1 & 1 \\ 1 & (1+x) & 1 \\ 1 & 1 & (1+x) \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$ (संक्रिया $C_1 \to C_1 + C_2 + C_3$ से)

$$a^2b^2c^2(3+x) \begin{vmatrix} 1 & 1 & 1 \\ 1 & (1+x) & 1 \\ 1 & 1 & (1+x) \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$ (संक्रिया $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$ से)

$$= \begin{vmatrix} 0 & -x & 0 \\ 0 & x & -x \\ 1 & 1 & 1+x \end{vmatrix} a^2b^2c^2(3+x) = a^2b^2c^2(3+x)\,x^2$$

Which is divisible by x^2 (जो कि x^2 से विभजित है)

7. If A is a non–singular matrix and A^T denotes the transpose of A, then:
यदि A एक व्युत्क्रमणीय आव्यूह है और A^T , A के परिवर्त को प्रदर्शित करता हो, तो –

(A) $|A| \neq |A^T|$ (B*) $|A.\ A^T| = |A|^2$
(C*) $|A^T.\ A| = |A^T|^2$ (D*) $|A|+|A^T| \neq 0$

Sol. $|A| + |A^T| = 2|A| \neq 0$

8. Let $f(x) = \begin{vmatrix} 2\sin x & \sin^2 x & 0 \\ 1 & 2\sin x & \sin^2 x \\ 0 & 1 & 2\sin x \end{vmatrix}$, then

(A) f(x) is independent of x (B*) $f'(\pi/2) = 0$

(C*) $\int_{-\pi/2}^{\pi/2} f(x)dx = 0$ (D*) tangent to the curve $y = f(x)$ at $x = 0$ is $y = 0$

यदि $f(x) = \begin{vmatrix} 2\sin x & \sin^2 x & 0 \\ 1 & 2\sin x & \sin^2 x \\ 0 & 1 & 2\sin x \end{vmatrix}$ हो, तो

(A) f(x), x से स्वतंत्र है। (B*) $f'(\pi/2) = 0$

(C*) $\int_{-\pi/2}^{\pi/2} f(x)dx = 0$ (D*) $x = 0$ पर वक्र $y = f(x)$ की स्पर्श रेखा $y = 0$ है।

Sol. $f(x) = \begin{vmatrix} 2\sin x & \sin^2 x & 0 \\ 1 & 2\sin x & \sin^2 x \\ 0 & 1 & 2\sin x \end{vmatrix}$ $= 2\sin x\,(4\sin^2 x - \sin^2 x) - \sin^2 x(2\sin x) = 6\sin^3 x - 2\sin^3 x$

$f(x) = 4\sin^3 x \Rightarrow f'(x) = 12\sin^2 x \cos x$

(B) $f'(\pi/2) = 12\sin^2(\pi/2)\cos(\pi/2) = 0$

(C) $f(-x) = -f(x)$ odd function

$\therefore \int_{-\pi/2}^{\pi/2} f(x)dx = 0$

(D) at $x = 0, y = 0 \Rightarrow \left(\frac{dy}{dx}\right)_{(0,0)} = 0$ tangent at $(0, 0)$

$y - 0 = \left(\frac{dy}{dx}\right)_{(0,0)}(x - 0) \Rightarrow y = 0$

Hindi. $f(x) = \begin{vmatrix} 2\sin x & \sin^2 x & 0 \\ 1 & 2\sin x & \sin^2 x \\ 0 & 1 & 2\sin x \end{vmatrix}$ $= 2\sin x\,(4\sin^2 x - \sin^2 x) - \sin^2 x(2\sin x) = 6\sin^3 x - 2\sin^3 x$

$f(x) = 4\sin^3 x \Rightarrow f'(x) = 12\sin^2 x \cos x$

(B) $f'(\pi/2) = 12\sin^2(\pi/2)\cos(\pi/2) = 0$

(C) $f(-x) = -f(x)$ विषम फलन $\therefore \int_{-\pi/2}^{\pi/2} f(x)dx = 0$

(D) at $x = 0, y = 0 \Rightarrow \left(\frac{dy}{dx}\right)_{(0,0)} = 0,$

(0, 0) पर स्पर्श रेखा का समीकरण $y - 0 = \left(\frac{dy}{dx}\right)_{(0,0)}(x - 0),\quad y = 0$

9. Let $f(x) = \begin{vmatrix} 1/x & \log x & x^n \\ 1 & -1/n & (-1)^n \\ 1 & a & a^2 \end{vmatrix}$, then (where $f^n(x)$ denotes n^{th} derivative of f(x))

(A*) $f^n(1)$ is independet of a
(B*) $f^n(1)$ is independet of n
(C) $f^n(1)$ depends on a and n
(D*) $y = a(x - f^n(1))$ represents a straight line through the origin

माना $f(x) = \begin{vmatrix} 1/x & \log x & x^n \\ 1 & -1/n & (-1)^n \\ 1 & a & a^2 \end{vmatrix}$ हो, तो (जहाँ $f^n(x)$, f(x) का nवाँ अवकलज है।)

(A*) $f^n(1)$, a से स्वतंत्र है।
(B*) $f^n(1)$, n से स्वतंत्र है।
(C) $f^n(1)$, a एवं n पर निर्भर करता हैं
(D*) $y = a(x - f^n(1))$ एक सरल रेखा को प्रदर्शित करता है जो मूल बिन्दु से गुजरती है।

Sol. $f(x) = \begin{vmatrix} 1/x & \log x & x^n \\ 1 & -1/n & (-1)^n \\ 1 & a & a^2 \end{vmatrix}$; $f'(x) = \begin{vmatrix} -1/x^2 & 1/x & nx^{n-1} \\ 1 & -1/n & (-1)^n \\ 1 & a & a^2 \end{vmatrix}$

$$f''(x) = \begin{vmatrix} 2/x^3 & -1/x^2 & n(n-1)x^{n-2} \\ 1 & -1/n & (-1)^n \\ 1 & a & a^2 \end{vmatrix} \quad ; \quad f^n(x) = \begin{vmatrix} (-1)^n \dfrac{n\,!}{x^{n+1}} & \dfrac{(-1)^{n-1}\,(n-1)\,!}{x^n} & n\,! \\ 1 & -1/n & (-1)^n \\ 1 & a & a^2 \end{vmatrix}$$

$$f^n(1) = \begin{vmatrix} (-1)^n n\,! & (-1)^{n-1}(n-1)! & n\,! \\ 1 & -1/n & (-1)^n \\ 1 & a & a^2 \end{vmatrix} = (-1)^n\, n! \begin{vmatrix} 1 & -1/n & (-1)^n \\ 1 & -1/n & (-1)^n \\ 1 & a & a^2 \end{vmatrix} = 0$$

and तथा $y = a\,(x - f^n(1))$

$y = ax$

10. Let A, B, C, D be real matrices such that $A^T = BCD$; $B^T = CDA$; $C^T = DAB$ and $D^T = ABC$ for the matrix $M = ABCD$, then find M^{2016} ?

माना कि A, B, C, D वास्तविक आव्यूह इस प्रकार है कि $A^T = BCD$; $B^T = CDA$; $C^T = DAB$ और $D^T = ABC$ तब आव्यूह के लिए $M = ABCD$ के लिए M^{2016} का मान है ?

(A) M (B*) M^2 (C) M^3 (D*) M^4

Sol. $M = ABCD = A(BCD) = AA^T$

$M^3 = (ABCD)\,(ABCD)\,(ABCD) = (ABC)\,(DAB)\,(CDA)\,(BCD) = D^T C^T B^T A^T = (BCD)^T A^T = AA^T = M.$

Similarly सामान्यतया $M^{9k} = M.$

11. Let **A** and **B** be two 2 × 2 matrix with real entries. If **AB** = **O** and tr(**A**) = tr(**B**) = 0 then

(A*) **A** and **B** are comutative w.r.t. operation of multiplication.

(B) **A** and **B** are not commutative w.r.t. operation of multiplication.

(C) **A** and **B** are both null matrices.

(D*) **BA** = 0

माना **A** तथा **B,** दो 2 × 2 का वास्तविक आव्यूह है यदि **AB** = **O** तथा tr(**A**) = tr(**B**) = 0 हो तो

(A*) गुणन संक्रिया के सापेक्ष **A** तथा **B** क्रमविनिमय है।

(B) गुणन संक्रिया के सापेक्ष **A** तथा **B** क्रमविनिमय नहीं है।

(C) **A** तथा **B** दोनों रिक्त आव्यूह है।

(D*) **BA** = 0

Sol. Let (माना) $\mathbf{A} = \begin{bmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{bmatrix}$ and (तथा) $\mathbf{B} = \begin{bmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{bmatrix} \Rightarrow \mathbf{AB} = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 - b_1a_2 \\ c_1a_2 - a_1c_2 & c_1b_2 + a_1a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow\ a_1a_2 + b_1c_2 = a_1b_2 - b_1a_2 = c_1a_2 - a_1c_2 = c_1b_2 + a_1a_2 = 0$

$\mathbf{BA} = \begin{bmatrix} a_2a_1 + b_2c_1 & a_2b_1 - b_2a_1 \\ c_2a_1 - a_2c_1 & c_2b_1 + a_2a_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

12. If $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, then

(A) | A | = 2 (B*) A is non-singular

(C*) Adj. A = $\begin{bmatrix} 1/2 & -1/2 & 0 \\ 0 & -1 & 1/2 \\ 0 & 0 & -1/2 \end{bmatrix}$ (D) A is skew symmetric matrix

यदि $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ हो, तो –

(A) | A | = 2 (B*) A व्युत्क्रमणीय हैं।

(C*) Adj. A = $\begin{bmatrix} 1/2 & -1/2 & 0 \\ 0 & -1 & 1/2 \\ 0 & 0 & -1/2 \end{bmatrix}$ (D) A विषम सममित मैट्रिक्स हैं।

Sol. adj A = $A^{-1}|A| = \frac{1}{2}\begin{vmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 1/2 & -1/2 & 0 \\ 0 & -1 & 1/2 \\ 0 & 0 & -1/2 \end{vmatrix}$

Here (यहाँ) $A^{-1} = \frac{\text{adj } A}{|A|}$, $|A^{-1}| = \frac{1}{|A|}$ and (तथा) $|A| = \frac{1}{2}$

13. If A and B are square matrices of order 3, then the true statement is/are (where I is unit matrix).
(A*) det (– A) = – det A
(B*) If AB is singular then atleast one of A or B is singular
(C) det (A + I) = 1 + det A
(D*) det (2A) = 2^3 det A

यदि A और B, 3 क्रम का वर्ग आव्यूह हो, तो निम्न में से सही कथन हैं (जहाँ I इकाई आव्यूह हैं) –
(A*) det (– A) = – det A
(B*) यदि AB अव्युत्क्रमणीय आव्यूह है तब A या B मे से कम से कम एक अव्युत्क्रमणीय है।
(C) det (A + I) = 1 + det A
(D*) det (2A) = 2^3 det A

Sol. det (–A) = $(-1)^n$ det (A) where n is order of square matrix.
(det (–A) = $(-1)^n$ det (A) जहाँ n क्रम वर्ग का आव्यूह है)

14. Let **M** be a 3 × 3 non-singular matrix with det(**M**) = 4. If $\mathbf{M}^{-1}$ adj(adj **M**) = $\mathbf{k}^2$I, then the value of **'k'** may be :
माना **M** एक 3 × 3 क्रम का व्युत्क्रमणीय आव्यूह है जहाँ det(**M**) = 4 यदि $\mathbf{M}^{-1}$ adj(adj **M**) = $\mathbf{k}^2$I तो **'k'** का मान हो सकता है–
(A*) +2 (B) 4 (C*) –2 (D) –4

Sol. $\mathbf{M}^{-1}$ adj(adj **M**) = $\mathbf{k}^2$I
Pre-multiplying by **M** **(M** से पूर्वगुणन करने पर**)**
adj(adj **M**) = $\mathbf{k}^2\mathbf{M}$
det(adj(adj M)) = det($\mathbf{k}^2\mathbf{M}$)
= $\mathbf{k}^6$ det (**M**)
(det **M**)^4 = $\mathbf{k}^6$ (det **M**)
(det **M**)^3 = $\mathbf{k}^6$
det **M** = $\mathbf{k}^2$ ⇒ $k^2 = \alpha$ ⇒ $k^2 = 4$

15. If AX = B where A is 3 × 3 and X and B are 3×1 matrices then which of the following is correct?
(A) If |A| = 0 then AX = B has infinite solutions
(B*) If AX = B has infinite solutions then |A| = 0
(C*) If (adj(A)) B = 0 and |A| ≠ 0 then AX = B has unique solution
(D*) If (adj(A)) B ≠ 0 & |A| = 0 then AX = B has no solution
यदि AX = B जहाँ A एक 3 × 3 क्रम का आव्यूह है और X और B, 3×1 क्रम का आव्यूह है तब निम्न मे से कौनसा सही है?
(A) यदि |A| = 0 तब AX = B के अनन्त हल है।
(B*) यदि AX = B के अनन्त हल है तब |A| = 0
(C*) यदि (adj(A)) B = 0 और |A| ≠ 0 तब AX = B अद्वितीय हल रखता है।
(D*) यदि (adj(A)) B ≠ 0 और |A| = 0 तब AX = B का कोई हल नही है।

Sol. Obvious

16✍_. Let M = $\begin{bmatrix} 0 & 1 & b \\ 1 & 2 & d \\ a & c & 1 \end{bmatrix}$ and adj M = $\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & e \end{bmatrix}$

than which of the following option is/are correct ?

(A*) a + b + c + d + e = 8 (B*) |M| = –2

(C) a = 3 (D) b = 3

माना M = $\begin{bmatrix} 0 & 1 & b \\ 1 & 2 & d \\ a & c & 1 \end{bmatrix}$ और adj M = $\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & e \end{bmatrix}$

तब निम्न मे से कौनसा विकल्प सही है?

(A*) a + b + c + d + e = 8 (B*) |M| = –2

(C) a = 3 (D) b = 3

Sol. adj(adj M) = |M| M

compare their element of 3rd row and 3rd column

$\Rightarrow$ |M| (1) = (6 – 8) $\Rightarrow$ |M| = –2

e = (0 × 2 – 1× 1) = –1

Now |M| × b = 2 – 6 $\Rightarrow$ b = 2

|M| × a = 24 – 30 $\Rightarrow$ a = 3

|M| × d = –(–2 + 8) $\Rightarrow$ d = 3

|M| × c = –(–3 + 5) $\Rightarrow$ c = 1 $\Rightarrow$ a + b + c + d + e = 8

Hindi. adj(adj M) = |M| m

3 पंक्ति और 3 स्तम्भ से तुलना करने पर

$\Rightarrow$ |M| (1) = (6 – 8) $\Rightarrow$ |M| = –2

e = (0 × 2 – 1× 1) = –1

अब |M| × b = 2 – 6 $\Rightarrow$ b = 2

|M| × a = 24 – 30 $\Rightarrow$ a = 3

|M| × d = –(–2 + 8) $\Rightarrow$ d = 3

|M| × c = –(–3 + 5) $\Rightarrow$ c = 1 $\Rightarrow$ a + b + c + d + e = 8

17✍_. Let $P = \begin{bmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ a & b & c \end{bmatrix}$, PP^T is diagonal matrix, Trace of $P^TAP = 49$, where in matrix $A = [a_{ij}]$,

$a_{11} + a_{22} = 0$, $a_{33} = 1$, then which of the following statement is/are correct ?

(A*) $|Det(P)| = 343$ (B*) $|a| + |b| + |c| = 11$

(C*) $Tr(PP^T) = 147$ (D) APP^T is diagonal matrix

माना $P = \begin{bmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ a & b & c \end{bmatrix}$, PP^T एक विकर्ण आव्यूह है P^TAP का अनुरेख $= 49$ जहाँ आव्यूह $A = [a_{ij}]$

$a_{11} + a_{22} = 0$, $a_{33} = 1$, तब निम्न में से कौनसा कथन गलत है –

(A*) $|Det(P)| = 343$ (B*) $|a| + |b| + |c| = 11$

(C*) $Tr(PP^T) = 147$ (D) APP^T विकर्ण आव्यूह

Sol. $\begin{bmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ a & b & c \end{bmatrix}\begin{bmatrix} 2 & 6 & a \\ 3 & 2 & b \\ 6 & -3 & c \end{bmatrix}$ = diagonal matrix $\Rightarrow$ $2a + 3b + 6c = 0,\ 6a + 2b - 3c = 0$

$\Rightarrow \quad \dfrac{a}{-21} = \dfrac{b}{42} = \dfrac{c}{-14} \quad \Rightarrow \quad a = -3\lambda,\ b = 6\lambda,\ c = -2\lambda$

Now trace of P^TAP = trace of PP^TA

$$= \text{trace of} \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49\lambda^2 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$= 49(a_{11} + a_{22}) + 49a_{33}\lambda^2 = 49 \Rightarrow 49\lambda^2 = 49 \Rightarrow \lambda = \pm 1$

18^. If $A = \begin{bmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{bmatrix}$, $B = \begin{bmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{bmatrix}$, $U = \begin{bmatrix} f \\ g \\ h \end{bmatrix}$, $V = \begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and AX = U has infinitely many solution. Then which of the following statement is/are correct
(A) BX = V has unique solution
(B*) BX = V has no unique solution
(C*) if $afd \neq 0$, then BX = V has no solution.
(D) if $afd \neq 0$, then BX = V has unique solution.

यदि $A = \begin{bmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{bmatrix}$, $B = \begin{bmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{bmatrix}$, $U = \begin{bmatrix} f \\ g \\ h \end{bmatrix}$, $V = \begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ और AX = U के अनन्त हल हो, तो निम्न में से कौनसा/कौनसे कथन सत्य हैं।

(A) BX = V का अद्वितीय हल है
(B*) BX = V का कोई हल नहीं है।
(C*) यदि $afd \neq 0$, तब BX = V का अद्वितीय हल है।
(D) यदि $afd \neq 0$, तब BX = V का अद्वितीय हल है।

Sol. Since AX = U has infinitely many solution
चूँकि AX = U के अनन्त हल है।

$$|A| = 0 \quad \begin{vmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{vmatrix} = 0$$

a(bc – db) + 1 (d – c) = 0
(d – c) (ab – 1) = 0
ab = 1, d = c

$$|A_1| = \begin{vmatrix} f & 1 & 0 \\ g & b & d \\ h & b & c \end{vmatrix} = 0 \qquad \Rightarrow \qquad c = d,\ g = h$$

$$|A_3| = \begin{vmatrix} a & 1 & f \\ 1 & b & g \\ 1 & b & h \end{vmatrix} = 0 \qquad \Rightarrow \qquad g = h$$

$$|A_2| = \begin{vmatrix} a & f & 0 \\ 1 & g & d \\ 1 & h & c \end{vmatrix} = 0 \qquad \Rightarrow \qquad g = h,\ c = d$$

g = h, c = d

BX = V

$$|B| = \begin{vmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{vmatrix} = 0$$

By (1) C_2 and C_3 are equal (1) से C_2 एवं C_3 बराबर है।

$$\therefore \quad |B| = 0 \quad ; \quad |B_1| = \begin{vmatrix} a^2 & 1 & 1 \\ 0 & d & c \\ 0 & g & h \end{vmatrix} = 0$$

Since (चूंकि) c = d and (एवं) g = h

$$|B_2| = \begin{vmatrix} a & a^2 & 1 \\ 0 & 0 & c \\ f & 0 & h \end{vmatrix} = a^2\,cf = a^2 df$$

$adf \neq 0 \qquad \Rightarrow \qquad |B_2| \neq 0$

$|B| = 0$ and (एवं) $|B_2| \neq 0$

BX = V has no solution BX = V का कोई हल नहीं है।

PART - IV : COMPREHENSION

भाग - IV : अनुच्छेद (COMPREHENSION)

Comprehension # 1

अनुच्छेद # 1

Let $\mathcal{A}$ be the set of all 3×3 symmetric matrices whose entries are 1,1,1,0,0,0,–1, –1, –1. B is one of the matrix in set $\mathcal{A}$ and

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad U = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad V = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

1. Number of such matrices B in set $\mathcal{A}$ is λ, then λ lies in the interval
(A*) (30, 40) (B) (38, 40) (C*) (34, 38) (D) (25, 35)

2. Number of matrices B such that equation BX = U has infinite solutions
(A*) is at least 6 (B) is not more than 10 (C*) lie between 8 to 16 (D) is zero.

3. The equation BX = V

(A*) is inconsistent for atleast 3 matrices B.
(B) is inconsistent for all matrices B.
(C*) is inconsistent for at most 12 matrices B.
(D) has infinite number of solutions for at least 3 matrices B.

Sol. **(1)** Let any symmetric matrix B = $\begin{bmatrix} a & x & y \\ x & b & z \\ y & z & c \end{bmatrix}$

so total matrices possible = $3! \times 3! = 36$.

(2) For homogeneous system infinite solution are possible if |B| = 0
$|B| = abc + 2xyz - by^2 - cx^2 - az^2$.
abc = xyz = 0 (one of a,b,c and one of x, y, z is zero)

if	a = 0	z = 0	$\Rightarrow$	|B| = 0	- 4 cases
	b = 0	y = 0	$\Rightarrow$	|B| = 0	- 4 cases
	c = 0	x = 0	$\Rightarrow$	|B| = 0	- 4 cases

so total cases are 12.
(3) BX = V is always inconsistent. when |B| = 0
so 12 cases

अनुच्छेद # 1

माना 𝒜 सभी 3×3 क्रम की सममित आव्यूहों का समुच्चय है, जिनके अवयव 1,1,1,0,0,0,–1, –1, –1 हैं। माना B समुच्चय 𝒜 का एक आव्यूह है और

X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ U = $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ V = $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

तो निम्नलिखित प्रश्नों के उत्तर दीजिए।

1*. यदि समुच्चय 𝒜 में इस प्रकार के आव्यूहो B की संख्या λ हैं, तो λ अन्तराल में स्थित है–
(A*) (30, 40) (B) (38, 40) (C*) (34, 38) (D) (25, 35)

2*. यदि समीकरण BX = U अनन्त हल रखता है, तो आव्यूहों B की संख्या होगी –
(A*) कम से कम 6 (B) 10 से अधिक नहीं
(C*) 8 से 16 के मध्य (D) शून्य है

3*. समीकरण BX = V
(A*) कम से कम तीन आव्यूह B के लिए अंसगत है।
(B) सभी आव्यूह B के लिए अंसगत है।
(C*) अधिकतम 12 आव्यूह B के लिए अंसगत है।
(D) कम से कम तीन आव्यूह B के लिए अनन्त हल विद्यमान है।

Sol. **(1)** माना सममित आव्यूह B = $\begin{bmatrix} a & x & y \\ x & b & z \\ y & z & c \end{bmatrix}$

अतः संभव आव्यूह की कुल संख्या = $3! \times 3!$
= 36.
(2) समघात समीकरण निकाय के अनन्त हल के लिए |B| = 0

$|B| = abc + 2xyz - by^2 - cx^2 - az^2$.
abc = xyz = 0 (a,b,c में से कोई एक तथा x, y, z में से कोई एक अवश्य शून्य होगा)
यदि a = 0 z = 0 $\Rightarrow$ |B| = 0 - 4 cases

b = 0   y = 0 $\Rightarrow$ |B| = 0   - 4 cases
c = 0   x = 0 $\Rightarrow$ |B| = 0   - 4 cases

अतः   कुल 12 आव्यूह है.

(3)   BX = V सदैव असंगत होगी यदि |B| = 0

अतः 12 आव्यूह

Comprehension # 2

Some special square matrices are defined as follows :

Nilpotent matrix : A square matrix A is said to be nilpotent (of order 2) if, $A^2 = O$. A square matrix is said to be nilpotent of order p, if p is the least positive integer such that $A^p = O$.

Idempotent matrix : A square matrix A is said to be idempotent if, $A^2 = A$.

e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an idempotent matrix.

Involutory matrix : A square matrix A is said to be involutory if $A^2 = I$, I being the identity matrix.

e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an involutory matrix.

Orthogonal matrix : A square matrix A is said to be an orthogonal matrix if $A'A = I = AA'$.

4. If A and B are two square matrices such that AB = A & BA = B, then A & B are

(A*) Idempotent matrices   (B) Involutory matrices
(C) Orthogonal matrices   (D) Nilpotent matrices

Sol. AB = A
Premultiplying by B
BAB = BA
BB = B
$B^2 = B$ $\Rightarrow$ B is idempotent
similarly on post mutliplying by A
$ABA = A^2$ $\Rightarrow$ $AB = A^2$
$A = A^2$ $\Rightarrow$ A is idempotent

5. If the matrix $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal, then

(A) $\alpha = \pm \frac{1}{\sqrt{2}}$ (B) $\beta = \pm \frac{1}{\sqrt{6}}$ (C) $\gamma = \pm \frac{1}{\sqrt{3}}$ (D*) all of these

Sol. For orthogonal matrix $AA' = I$

$$\Rightarrow \begin{vmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{vmatrix} \begin{vmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \Rightarrow \begin{vmatrix} 4\beta^2+\gamma^2 & 2\beta^2-\gamma^2 & -2\beta^2+\gamma^2 \\ 2\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 \\ -2\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$\Rightarrow$ $4\beta^2 + \gamma^2 = 1,\ 2\beta^2 - \gamma^2 = 0,\ -2\beta^2 + \gamma^2 = 0,\ \alpha^2 - \beta^2 - \gamma^2 = 0,\ \alpha^2 + \beta^2 + \gamma^2 = 1$

$\therefore$ $\alpha = \pm\frac{1}{\sqrt{2}},\ \beta = \pm\frac{1}{\sqrt{6}},\ \gamma = \pm\frac{1}{\sqrt{3}}$

6. The matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is

(A) idempotent matrix   (B) involutory matrix
(C*) nilpotent matrix   (D) none of these

Sol. $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow A^2$ is nilpotent

अनुच्छेद # 2

कुछ विशेष वर्ग आव्यूह निम्नप्रकार परिभाषित है

शून्यभावी आव्यूह (Nilpotent matrix) : यदि कोई वर्ग आव्यूह A इस प्रकार है कि $A^2 = O$ हो, तो आव्यूह A को शून्यभावी आव्यूह (क्रम 2) कहते हैं। एक वर्ग आव्यूह p क्रम की शून्यभावी आव्यूह कहा जाता है यदि p सबसे कम धनात्मक पूर्णांक इस प्रकार हो कि $A^p = O$.

वर्गसम आव्यूह (Idempotent matrix) : एक वर्ग आव्यूह A को वर्गसम कहा जाता है यदि $A^2 = A$.

जैसे $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ एक वर्गसम आव्यूह हैं।

अन्तर्वलनीय आव्यूह (Involutory matrix) : एक वर्ग आव्यूह A को अन्तर्वलनीय कहा जाता है यदि $A^2 = I$, I इकाई आव्यूह है।

जैसे $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ एक अन्तर्वलनीय आव्यूह हैं।

लाम्बिक आव्यूह (Orthogonal matrix) : एक वर्ग आव्यूह A को लाम्बिक आव्यूह कहा जाता हैं

4. यदि A और B दो वर्ग आव्यूह इस प्रकार है कि AB = A एवं BA = B हो, तो A और B हैं –

(A*) वर्गसम आव्यूह (B) अन्तर्वलनीय आव्यूह (C) लाम्बिक आव्यूह (D) शून्यभावी आव्यूह

Sol. AB = A

B से पूर्वगुणन करने पर

BAB = BA

BB = B

$B^2 = B \Rightarrow$ B वर्गसम है

इसी प्रकार A से अग्रगुणज करने पर

$ABA = A^2$

$AB = A^2$

$A = A^2 \Rightarrow$ A वर्गसम आव्यूह है

5. यदि आव्यूह $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ लाम्बिक आव्यूह हो, तो–

(A) $\alpha = \pm \frac{1}{\sqrt{2}}$ (B) $\beta = \pm \frac{1}{\sqrt{6}}$ (C) $\gamma = \pm \frac{1}{\sqrt{3}}$ (D*) उपरोक्त सभी

Sol. लाम्बिक आव्यूह के लिए $AA' = I$

$$\Rightarrow \begin{vmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{vmatrix} \begin{vmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \Rightarrow \begin{vmatrix} 4\beta^2+\gamma^2 & 2\beta^2-\gamma^2 & -2\beta^2+\gamma^2 \\ 2\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 \\ -2\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$\Rightarrow \quad 4\beta^2 + \gamma^2 = 1,\ 2\beta^2 - \gamma^2 = 0,\ -2\beta^2 + \gamma^2 = 0,\ \alpha^2 - \beta^2 - \gamma^2 = 0,\ \alpha^2 + \beta^2 + \gamma^2 = 1$

$\therefore \quad \alpha = \pm\frac{1}{\sqrt{2}},\ \beta = \pm\frac{1}{\sqrt{6}},\ \gamma = \pm\frac{1}{\sqrt{3}}$

6. आव्यूह $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ है –

(A) वर्गसम आव्यूह (B) अन्तर्वलनीय आव्यूह
(C*) शून्यभावी आव्यूह (D) इनमें से कोई नहीं

Sol. $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow A^2$ is nilpotent

Exercise-3

✍ **Marked questions are recommended for Revision.**
✍ चिन्हित प्रश्न दोहराने योग्य प्रश्न है।

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

भाग - I : JEE (ADVANCED) / IIT-JEE (पिछले वर्षो) के प्रश्न

* **Marked Questions may have more than one correct option.**
* चिन्हित प्रश्न एक से अधिक सही विकल्प वाले प्रश्न है -

Comprehension (Q. No. 1 to 3)

Let p be an odd prime number and T_p be the following set of 2×2 matrices :

$$T_p = \left\{A = \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a,\ b,\ c \in \{0,\ 1,\ 2,\ldots\ldots,p-1\}\right\}$$

1. The number of A in T_p such that A is either symmetric or skew-symmetric or both, and det (A) divisible by p is **[IIT-JEE 2010, Paper-1, (3, –1), 84]**

(A) $(p-1)^2$ (B) $2(p-1)$ (C) $(p-1)^2 + 1$ (D*) $2p - 1$

Sol. $A = \begin{bmatrix} a & b \\ c & a \end{bmatrix}$ where $a, b, c \in \{0,1,2,\ldots.p-1\}$

Case- I A is symmetric matrix $\Rightarrow$ $b = c$

$\Rightarrow$ $\det(A) = a^2 - b^2$ is divisible by p $\Rightarrow$ $(a-b)(a+b)$ is divisible by p

(a) $a - b$ is divisible by p if $a = b$, then 'p' cases are possible

(b) $a + b$ is divisible by p if $a + b = p$, then $\dfrac{(p-1)}{2} \times 2 = (p-1)$ cases are possible

Case - II

A is skew symmetric matrix

if $a = 0$, $b + c = 0$, then $\det(A) = b^2 \Rightarrow b^2$ can never be divisible by p. So No case is possible

Total number of A is possible $= 2p - 1$

2. The number of A in T_p such that the trace of A is not divisible by p but det (A) is divisible by p is
[Note : The trace of matrix is the sum of its diagonal entries.] **[IIT-JEE 2010, Paper-1, (3, –1), 84]**

(A) $(p-1)(p^2 - p + 1)$ (B) $p^3 - (p-1)^2$
(C*) $(p-1)^2$ (D) $(p-1)(p^2-2)$

Sol. $a^2 - bc \div p$ $\Rightarrow$ a can be chosen in $p - 1$ ways $(a \neq 0)$

Let a be 4 & p = 5 $\Rightarrow$ so $a^2 = 16$ and hence bc should be chosen such that $a^2 - bc \div p$

Now b can be chosen in $p - 1$ ways and c in only 1 (one be is chosen)

Explanation : If $b = 1 \Rightarrow c = 1$; $b = 2 \Rightarrow c = 3$

$b = 3 \Rightarrow c = 2$; $b = 4 \Rightarrow c = 4$

Hence a can be chosen in $p - 1$ ways

and then b can be chosen in $p - 1$ ways

c can be chosen in 1 ways

so $(p-1)^2$

3. The number of A in T_p such that det (A) is not divisible by p is **[IIT-JEE 2010, Paper-1, (3, –1), 84]**

(A) $2p^2$ (B) $p^3 - 5p$ (C) $p^3 - 3p$ (D*) $p^3 - p^2$

Sol. As = Total cases – ($a \neq 0$ and |A| is divisible by p) – ($a = 0$ and |A| is divisible by p)

$= p^3 - (p-1)^2 - (2p-1) = p^3 - p^2 \Rightarrow -bc$ p since b & c both we coprime to p

$\Rightarrow$ one of them must be zero.

If $b = 0 \Rightarrow$ c can be chosen in $\{0,1,....p-1\}$
If $c = 0 \Rightarrow$ b can be chosen in $\{0,1,....p-1\}$

अनुच्छेद (प्रश्न 1 - 3 के लिए)

माना कि p एक विषम अभाज्य संख्या (odd prime number) है तथा T_p, 2×2 आव्यूहों का निम्न समुच्चय है :

$$T_p = \left\{ A = \begin{bmatrix} a & b \\ c & a \end{bmatrix} : \ a, \ b, \ c \ \in \{0, \ 1, \ 2, ...,p-1\} \right\}$$

1. T_p में ऐसे A की संख्या जो सममित, विषम सममित अथवा दोनों प्रकार के हों, एवं जिनके लिए det (A), p से विभाज्य हो, निम्न है– **[IIT-JEE 2010, Paper-1, (3, –1), 84]**

(A) $(p-1)^2$ (B) $2(p-1)$ (C) $(p-1)^2 + 1$ (D*) $2p - 1$

Sol. $A = \begin{bmatrix} a & b \\ c & a \end{bmatrix}$ जहाँ $a, b, c \in \{0,1,2,....p-1\}$

Case- I A सममित आव्यूह है $\Rightarrow$ $b = c$

$\Rightarrow$ $\det(A) = a^2 - b^2$, p से विभाजित है $\Rightarrow$ $(a-b)(a+b)$, p से विभाजित है

(a) $a - b$, p से विभाजित है यदि $a = b$, तब संभावित स्थितियाँ 'p' है

(b) $a + b$, p से विभाजित है यदि $a + b = p$, तब संभावित स्थितियाँ $\dfrac{(p-1)}{2} \times 2 = (p-1)$ है

Case - II

A विषम सममित आव्यूह है

यदि $a = 0$, $b + c = 0$, तब $\det(A) = b^2$ $\Rightarrow b^2$, p से कभी विभाजित नहीं होगा कोई संभावित स्थिति नहीं

A की कुल संभावित स्थिति $= 2p - 1$

2. T_p में ऐसे A की संख्या जिनके लिए trace (A), p से विभाज्य नहीं है, परन्तु det (A), p से विभाज्य है, निम्न है–
[**नोट** : trace (A), A के विकर्णीय प्रविष्टियों का योग होता है] **[IIT-JEE 2010, Paper-1, (3, –1), 84]**

(A) $(p-1)(p^2 - p + 1)$ (B) $p^3 - (p-1)^2$
(C*) $(p-1)^2$ (D) $(p-1)(p^2 - 2)$

Sol. $a^2 - bc \div p$

a को $p - 1$ तरीकों से चुना जाता है $(a \neq 0)$

माना $a = 4$ और $p = 5$

इसलिए $a^2 = 16$ अतः bc इस प्रकार चुना जाता है कि $a^2 - bc \div p$

अब b को $p - 1$ तरीकों से चुना जाता है और c को केवल 1 तरीके से चुना जाता है

व्याख्या: यदि $b = 1 \Rightarrow c = 1$; $b = 2 \Rightarrow c = 3$

$b = 3 \Rightarrow c = 2$; $b = 4 \Rightarrow c = 4$

अतः a, $p - 1$ तरीकों से चुना जा सकता है

और तब b, $p - 1$ तरीकों से चुना जा सकता है

c एक तरीके से चुना जा सकता है

इसलिए $(p-1)^2$

3. T_p में ऐसे A की संख्या जिनके लिए det (A), p से विभाज्य नहीं है, निम्न है–**[IIT-JEE 2010, Paper-1, (3, –1), 84]**

(A) $2p^2$ (B) $p^3 - 5p$ (C) $p^3 - 3p$ (D*) $p^3 - p^2$

Sol. कुल स्थितियाँ – ($a \neq 0$ तथा $|A|$, p से विभाजित है) – ($a = 0$ तथा $|A|$, p से विभाजित है)

$= p^3 - (p-1)^2 - (2p-1) = p^3 - p^2$

$-bc$ p चूँकि b और c दोनों p के सहअभाज्य है

$\Rightarrow$ उनमें से एक को शून्य होना है

यदि $b = 0 \Rightarrow$ तो c को $\{0,1,....p-1\}$ से चुना जा सकता है

यदि $c = 0 \Rightarrow$ तो b को $\{0,1,....p-1\}$ से चुना जा सकता है

4. The number of all possible values of θ, where $0 < \theta < \pi$, for which the system of equations

$$(y + z) \cos 3\theta = (xyz) \sin 3\theta$$
$$x \sin 3\theta = \frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z}$$
$$(xyz) \sin 3\theta = (y + 2z) \cos 3\theta + y \sin 3\theta$$

have a solution (x_0, y_0, z_0) with $y_0 z_0 \neq 0$, is **[IIT-JEE 2010, Paper-1, (3, 0), 84]**

$0 < \theta < \pi$ को सन्तुष्ट करने वाले θ के सभी सम्भावित मानों की संख्या, जिनके लिए समीकरण समूह

$$(y + z) \cos 3\theta = (xyz) \sin 3\theta$$
$$x \sin 3\theta = \frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z}$$
$$(xyz) \sin 3\theta = (y + 2z) \cos 3\theta + y \sin 3\theta$$

का एक हल (x_0, y_0, z_0) है, जहाँ $y_0 z_0 \neq 0$, है

Ans. 3

Sol. Let xyz = t

$t \sin 3\theta - y \cos 3\theta - z \cos 3\theta = 0$ ....(1)

$t \sin 3\theta - 2y \sin 3\theta - 2z \cos 3\theta = 0$ ....(2)

$t \sin 3\theta - y (\cos 3\theta + \sin 3\theta) - 2z \cos 3\theta = 0$ ....(3)

$y_0 . z_0 \neq 0$ hence homogeneous equation has non-trivial solution

$$D = \begin{vmatrix} \sin 3\theta & -\cos 3\theta & -\cos 3\theta \\ \sin 3\theta & -2\cos 3\theta & -2\cos 3\theta \\ \sin 3\theta & -(\cos 3\theta + \sin 3\theta) & -2\cos 3\theta \end{vmatrix} = 0$$

$\Rightarrow$ $\sin 3\theta \cos 3\theta (\sin 3\theta - \cos 3\theta) = 0$ $\Rightarrow$ $\sin 3\theta = 0$ or $\cos 3\theta = 0$ or $\tan 3\theta = 1$

Case - I $\sin 3\theta = 0$

From equation (2)

$z = 0$ not possible

Case - II $\cos 3\theta = 0, \sin 3\theta \neq 0$

$t. \sin 3\theta = 0$ $\Rightarrow$ $t = 0$ $\Rightarrow$ $x = 0$

From equation (2)

$y = 0$ not possible

Case- III $\tan 3\theta = 1$ $\Rightarrow$ $3\theta = n\pi + \frac{\pi}{4}, n \in I$

$\Rightarrow$ $x. y. z \sin 3\theta = 0$ $\Rightarrow$ $\theta = \frac{n\pi}{3} + \frac{\pi}{12}, n \in I$

$\Rightarrow$ $x = 0, \sin 3\theta \neq 0$ $\Rightarrow$ $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}$

Hence 3 solutions

Hindi माना xyz = t

$t \sin 3\theta - y \cos 3\theta - z \cos 3\theta = 0$ ....(1)

$t \sin 3\theta - 2y \sin 3\theta - 2z \cos 3\theta = 0$ ....(2)

$t \sin 3\theta - y (\cos 3\theta + \sin 3\theta) - 2z \cos 3\theta = 0$ ....(3)

$y_0 . z_0 \neq 0$ अतः समघात समीकरण के अशून्य हल है

$$D = \begin{vmatrix} \sin 3\theta & -\cos 3\theta & -\cos 3\theta \\ \sin 3\theta & -2\cos 3\theta & -2\cos 3\theta \\ \sin 3\theta & -(\cos 3\theta + \sin 3\theta) & -2\cos 3\theta \end{vmatrix} = 0$$

$\Rightarrow$ $\sin 3\theta \cos 3\theta (\sin 3\theta - \cos 3\theta) = 0$ $\Rightarrow$ $\sin 3\theta = 0$ या $\cos 3\theta = 0$ या $\tan 3\theta = 1$

Case - I $\sin 3\theta = 0$

समीकरण (2) से

$z = 0$ संभव नहीं है

Case - II $\cos 3\theta = 0, \sin 3\theta \neq 0$

$t.\ \sin 3\theta = 0$ $\Rightarrow$ $t = 0$ $\Rightarrow$ $x = 0$

समीकरण (2) से

$y = 0$ संभव नहीं है

Case- III $\tan 3\theta = 1$ $3\theta = n\pi + \frac{\pi}{4}, n \in I$

$\Rightarrow$ $x.\ y.\ z \sin 3\theta = 0$ $\Rightarrow$ $\theta = \frac{n\pi}{3} + \frac{\pi}{12}, n \in I$

$\Rightarrow$ $x = 0, \sin 3\theta \neq 0$ $\Rightarrow$ $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}$

अतः 3 हल

5. Let k be a positive real number and let **[IIT-JEE 2010, Paper-2, (3, 0), 79]**

$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}$. If det (adj A) + det (adj B) = 10^6, then [k] is equal to

(**Note** : adj M denotes the adjoint of a square matrix M and [k] denotes the largest integer less than or equal to k].

माना कि k एक धनात्मक वास्तविक संख्या है तथा

$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix}$ एवं $B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}$. यदि det (adj A) + det (adj B) = 10^6, तो [k] का मान है

(**नोट** : adj M किसी वर्ग आव्यूह M का adjoint तथा [k], अधिकतम पूर्णांक जो k से कम या k के समान है, को प्रदर्शित करता है।)

Ans. 4

Sol. $\det(A) = \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{vmatrix}$ $C_2 \to C_2 - C_3$

$= \begin{vmatrix} 2k-1 & 0 & 2\sqrt{k} \\ 2\sqrt{k} & 1+2k & -2k \\ -2\sqrt{k} & 2k+1 & -1 \end{vmatrix}$ $R_2 \to R_2 - R_3$

$= \begin{vmatrix} 2k-1 & 0 & 2\sqrt{k} \\ 4\sqrt{k} & 0 & 1-2k \\ -2\sqrt{k} & 2k+1 & -1 \end{vmatrix} = (2k+1)^3$

$\because$ B is a skew-symmetric matrix of odd order therefore det(B) = 0

Now det (adj A) + det (adj B) = 10^6

$\Rightarrow$ $\{(2k + 1)^3\}^2 + 0 = 10^6$ $\Rightarrow$ 2k + 1 = 10, as k > 0 $\Rightarrow$ k = 4.5 $\Rightarrow$ [k] = 4

Hindi $\det(A) = \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{vmatrix}$ $\quad C_2 \to C_2 - C_3$

$$= \begin{vmatrix} 2k-1 & 0 & 2\sqrt{k} \\ 2\sqrt{k} & 1+2k & -2k \\ -2\sqrt{k} & 2k+1 & -1 \end{vmatrix} \quad R_2 \to R_2 - R_3$$

$$= \begin{vmatrix} 2k-1 & 0 & 2\sqrt{k} \\ 4\sqrt{k} & 0 & 1-2k \\ -2\sqrt{k} & 2k+1 & -1 \end{vmatrix} = (2k+1)^3$$

$\because$ B विषम क्रम की विषम सममित आव्यूह है इसलिए det(B) = 0

अब det (adj A) + det (adj B) = 10^6

$\Rightarrow$ $\{(2k + 1)^3\}^2 + 0 = 10^6$ $\Rightarrow$ 2k + 1 = 10,

क्योंकि k > 0 $\Rightarrow$ k = 4.5 $\Rightarrow$ [k] = 4

6. Let M and N be two 2n × 2n non-singular skew-symmetric matrices such that MN = NM. If P^T denotes the transpose of P, then $M^2 N^2 (M^T N)^{-1} (MN^{-1})^T$ is equal to

(A) M^2 (B) $-N^2$ (C*) $-M^2$ (D) MN

मान लीजिए M और N दो ऐसे 2n × 2n व्युत्क्रमणीय और विषम सममित आव्यूह (non-singular skew-symmetric matrices) है, जो MN = NM को संतुष्ट करते है। यदि P^T, आव्यूह P के परिवर्त (transpose) आव्यूह को प्रदर्शित करता है, तो

$M^2 N^2 (M^T N)^{-1} (MN^{-1})^T$ निम्न में से किसके बराबर है– **[IIT-JEE 2011, Paper-1, (4, 0), 80]**

(A) M^2 के (B) $-N^2$ के (C*) $-M^2$ के (D) MN के

Ans. C (In JEE this question was bonus because in JEE instead of 2n × 2n, 3 × 3 was given and we know that there is no non-singular 3×3 skew symmetric matrix).

Ans. C (JEE में यह प्रश्न बोनस था क्योंकि JEE में 2n × 2n की जगह 3 × 3 दिया गया था एवं हम जानते है कि 3 × 3 क्रम का व्युत्क्रमणीय और विषम सममित आव्यूह (non-singular skew-symmetric matrices) सम्भव नहीं है)

Sol. **Data inconsistent**

A 3 × 3 non-singular matrix cannot be skew-symmetric

However considering M, N matrices as even order, we obtain correct answer.

$M^2 N^2 (M^T N)^{-1} (MN^{-1})^T = M^2N^2 N^{-1} (M^T)^{-1} (N^{-1})^T M^T$ $\Rightarrow$ $-M^2 N^2 N^{-1} M^{-1} N^{-1} M$

$\Rightarrow$ $-M^2 NM^{-1} N^{-1} M$ $\Rightarrow$ $-MNN^{-1} M$ $\Rightarrow$ $-M^2$

Hindi **आँकड़े अपर्याप्त है**

एक 3 × 3 व्युत्क्रमणीय आव्यूह विषम सममित आव्यूह नहीं हो सकती है।

फिर भी M और N आव्यूह को सम क्रम की लेने पर हम सही उत्तर प्राप्त करते है।

$M^2 N^2 (M^T N)^{-1} (MN^{-1})^T = M^2N^2 N^{-1} (M^T)^{-1} (N^{-1})^T M^T$ $\Rightarrow$ $-M^2 N^2 N^{-1} M^{-1} N^{-1} M$

$\Rightarrow$ $-M^2 NM^{-1} N^{-1} M$ $\Rightarrow$ $-MNN^{-1} M$ $\Rightarrow$ $-M^2$

Comprehension (7 to 9)

Let a, b and c be three real numbers satisfying

$$[a\ b\ c]\begin{bmatrix}1 & 9 & 7\\ 8 & 2 & 7\\ 7 & 3 & 7\end{bmatrix} = [0\ 0\ 0] \qquad \text{...........(E)}$$

माना कि a, b और c ऐसी तीन वास्तविक संख्याऐं है जो

$$[a\ b\ c]\begin{bmatrix}1 & 9 & 7\\ 8 & 2 & 7\\ 7 & 3 & 7\end{bmatrix} = [0\ 0\ 0] \qquad \text{...........(E)}$$

को संतुष्ट करती हैं। **[IIT-JEE 2011, Paper-1, (3, –1), 80]**

7. If the point P(a, b, c), with reference to (E), lies on the plane $2x + y + z = 1$, then the value of $7a + b + c$ is

यदि समीकरण (E) के संदर्भ में, बिन्दु P(a, b, c) समतल $2x + y + z = 1$ पर स्थित है, तो $7a + b + c$ का मान है–

(A) 0 (B) 12 (C) 7 (D*) 6

Ans. (D)

8. Let ω be a solution of $x^3 - 1 = 0$ with Im (ω) > 0. if a = 2 with b and c satisfying (E), then the value of $\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c}$ is equal to

माना कि ω, समीकरण $x^3 - 1 = 0$ का हल है, जहाँ Im (ω) > 0 है। यदि a = 2 और संगत संख्याएं b और c समीकरण (E) को संतुष्ट करती हैं तो $\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c}$ का मान है–

(A*) – 2 (B) 2 (C) 3 (D) – 3

Ans. (A)

9. Let b = 6, with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^n$ is

यदि b = 6 और संगत संख्याऐं a और c समीकरण (E) को संतुष्ट करती हों, और α , β द्विघातीय समीकरण $ax^2 + bx + c = 0$ के मूल हैं, तो $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^n$ का मान है–

(A) 6 (B*) 7 (C) $\frac{6}{7}$ (D) ∞

Ans. (B)

Sol. (Q.No. 7 to9)

$a + 8b + 7c = 0$ .... (i)

$9a + 2b + 3c = 0$ .... (ii)

$a + b + c = 0$ .... (iii)

$$\Delta = \begin{vmatrix}1 & 8 & 7\\ 9 & 2 & 3\\ 1 & 1 & 1\end{vmatrix} = 1.(-1) - 8(6) + 7(7) = 0$$

Let $c = \lambda$

$\therefore\ a + 8b = -7\lambda \quad \Rightarrow \quad a + b = -\lambda \quad \Rightarrow \quad b = \frac{-6}{7}\lambda\ \&\ a = \frac{-\lambda}{7}$

$\therefore\ (a, b, c) \equiv \left(\frac{-\lambda}{7}, \frac{-6\lambda}{7}, \lambda\right)$ where $\lambda \in R$

7. P(a, b, c) lies on the plane $2x + y + z = 1$

$\therefore \quad \frac{-2\lambda}{7} - \frac{6\lambda}{7} + \lambda = 1 \quad \Rightarrow \quad \frac{-\lambda}{7} = 1 \quad \Rightarrow \quad \lambda = -7$

$\therefore \quad 7a + b + c = 7 + 6 - 7 = 6$

8. $a = 2 \quad \Rightarrow \quad \lambda = -14$

$\therefore \quad b = 12 \quad \& \; c = -14$

Now $= \frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c} = \frac{3}{\omega^2} + \frac{1}{\omega^{12}} + 3.\omega^{14} \;\; 3\omega + 1 + 3\omega^2 = 3(\omega + \omega^2) + 1 = -2$

9. $b = 6 \quad \Rightarrow \quad \lambda = -7$

$\Rightarrow a = 1 \;\; \& \;\; c = -7$

now $\quad ax^2 + bx + c = 0 \quad \Rightarrow \quad x^2 + 6x - 7 = 0 \quad \Rightarrow \quad x = -7, 1$

$\therefore \quad \sum_{n=0}^{\infty}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^n = \sum_{n=0}^{\infty}\left(\frac{6}{7}\right)^n = 1 + \frac{6}{7} + \left(\frac{6}{7}\right)^2 + \ldots\ldots\; \infty = \frac{1}{1 - \frac{6}{7}} = 7$

Hindi **(Q.No. 7 to 9)**

$a + 8b + 7c = 0 \qquad$ .... (i)

$9a + 2b + 3c = 0 \qquad$ .... (ii)

$a + b + c = 0 \qquad$ ....(iii)

$$\Delta = \begin{vmatrix} 1 & 8 & 7 \\ 9 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 1.(-1) - 8(6) + 7(7) = 0$$

माना $C = \lambda$

$\therefore \; a + 8b = -7\lambda$

$a + b = -\lambda \quad \Rightarrow \quad b = \frac{-6}{7}\lambda$ तथा $a = \frac{-\lambda}{7}$

$\therefore \quad (a, b, c) \equiv \left(\frac{-\lambda}{7} \;,\; \frac{-6\lambda}{7} \;,\; \lambda\right)$ जहाँ $\lambda \in R$

7. $P(a, b, c)$ समतल $2x + y + z = 1$ पर स्थित है।

$\therefore \quad \frac{-2\lambda}{7} - \frac{6\lambda}{7} + \lambda = 1 \quad \Rightarrow \quad \frac{-\lambda}{7} = 1 \quad \Rightarrow \quad \lambda = -7$

$\therefore \quad 7a + b + c = 7 + 6 - 7 = 6$

8. $a = 2 \quad \Rightarrow \quad \lambda = -14$

$\therefore \quad b = 12$ तथा $c = -14$

अब $\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c} = \frac{3}{\omega^2} + \frac{1}{\omega^{12}} + 3.\omega^{14} = 3\omega + 1 + 3\omega^2 = 3(\omega + \omega^2) + 1 = -2$

9. $b = 6 \quad \Rightarrow \quad \lambda = -7 \quad \Rightarrow \quad a = 1$ तथा $c = -7$

अब $ax^2 + bx + c = 0 \quad \Rightarrow \quad x^2 + 6x - 7 = 0 \quad \Rightarrow \quad x = -7, 1$

$\therefore \quad \sum_{n=0}^{\infty}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^n = \sum_{n=0}^{\infty}\left(\frac{6}{7}\right)^n = 1 + \frac{6}{7} + \left(\frac{6}{7}\right)^2 + \ldots\ldots\; \infty = \frac{1}{1 - \frac{6}{7}} = 7$

10. Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form, $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$

where each of a, b and c is either ω or ω^2. Then the number of distinct matrices in the set S is

मान लीजिए $\omega \neq 1$ इकाई का घनमूल है और S, $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$ रूप वाली व्युत्क्रमणीय आव्यूहों (non-singular matrices)

[IIT-JEE 2011, Paper-2, (3, –1), 80]

का समुच्चय है। जहाँ a, b और c में से प्रत्येक या तो ω है, या ω^2 तब समुच्चय S के विभिन्न आव्यूहों की संख्या निम्न होगी–

(A*) 2 (B) 6 (C) 4 (D) 8

Ans. **(A)**

Sol. $a, b, c \in \{\omega, \omega^2\}$

Let $A = \begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix} \Rightarrow |A| = 1 - (a + c)\,\omega + ac\omega^2$

Now $|A|$ will be non-zero only when $a = c = \omega$

$\therefore$ $(a, b, c) \equiv (\omega, \omega, \omega)$ or $(\omega, \omega^2, \omega)$

$\therefore$ number of non singular matrices = 2

Hindi $a, b, c \in \{\omega, \omega^2\}$

माना $A = \begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix} \Rightarrow |A| = 1 - (a + c)\,\omega + ac\omega^2$

अब $|A|$ अशून्य होगा

केवल जब $a = c = \omega$

$\therefore$ $(a, b, c) \equiv (\omega, \omega, \omega)$ or $(\omega, \omega^2, \omega)$

$\therefore$ व्युत्क्रमणीय आव्यूहों की संख्या = 2

11. Let M be a 3×3 matrix satisfying

$M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, $M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, and $M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$. Then the sum of the diagonal entries of M is

माना कि M एक 3×3 आव्यूह है जो निम्न समीकरणों **[IIT-JEE 2011, Paper-2, (4, 0), 80]**

$M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, $M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, तथा $M = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$. को संतुष्ट करता है। तब M के विकर्ण के अवयवों का योग ज्ञात कीजिये।

Ans. 9

Sol. Let $M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

then $a_{12} = -1$, $a_{11} - a_{12} = 1 \Rightarrow a_{11} = 0$, $a_{11} + a_{12} + a_{13} = 0 \Rightarrow a_{13} = 1$

$a_{22} = 2$, $a_{21} - a_{22} = 1 \Rightarrow a_{21} = 3$, $a_{21} + a_{22} + a_{23} = 0 \Rightarrow a_{23} = -5$

$a_{32} = 3$, $a_{31} - a_{32} = -1 \Rightarrow a_{31} = 2$, $a_{31} + a_{32} + a_{33} = 12 \Rightarrow a_{33} = 7$

Hence sum of diagonal of M is $= a_{11} + a_{22} + a_{33} = 0 + 2 + 7 = 9$

Hindi माना $M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

तब $a_{12} = -1$, $a_{11} - a_{12} = 1 \Rightarrow a_{11} = 0$, $a_{11} + a_{12} + a_{13} = 0 \Rightarrow a_{13} = 1$

$a_{22} = 2$, $a_{21} - a_{22} = 1 \Rightarrow a_{21} = 3$, $a_{21} + a_{22} + a_{23} = 0 \Rightarrow a_{23} = -5$

$a_{32} = 3$, $\quad a_{31} - a_{32} = -1 \Rightarrow a_{31} = 2$, $\quad a_{31} + a_{32} + a_{33} = 12 \Rightarrow a_{33} = 7$

तब M के विकर्णो का योगफल $= a_{11} + a_{22} + a_{33} = 0 + 2 + 7 = 9$

12. Let $P = [a_{ij}]$ be a 3×3 matrix and let $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j}a_{ij}$ for $1 \le i, j \le 3$. If the determinant of P is 2, then the determinant of the matrix Q is **[IIT-JEE 2012, Paper-1, (3, –1), 70]**

माना कि $P = [a_{ij}]$ एक 3×3 आव्यूह (matrix) है और $Q = [b_{ij}]$, जहाँ $b_{ij} = 2^{i+j}a_{ij}$ जब $1 \le i, j \le 3$ है। यदि P के सारणिक (determinant) का मान 2 है तो आव्यूह Q के सारणिक का मान निम्न है **[IIT-JEE 2012, Paper-1, (3, –1), 70]**

(A) 2^{10} (B) 2^{11} (C) 2^{12} (D*) 2^{13}

Sol. Ans (D)

Given $P = [a_{ij}]_{3\times3}$ $\quad b_{ij} = 2^{i+j}$ aij

$Q = [b_{ij}]_{3\times3}$

$$P = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad |P| = 2 \quad ; \quad Q = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 4a_{11} & 8\ a_{12} & 16\ a_{13} \\ 8a_{21} & 16\ a_{22} & 32\ a_{23} \\ 16a_{31} & 32\ a_{32} & 64\ a_{33} \end{bmatrix}$$

$$\text{Determinant of Q} = \begin{vmatrix} 4a_{11} & 8\ a_{12} & 16\ a_{13} \\ 8a_{21} & 16\ a_{22} & 32\ a_{23} \\ 16a_{31} & 32\ a_{32} & 64\ a_{33} \end{vmatrix} = 4 \times 8 \times 16 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & 2\ a_{22} & 2\ a_{23} \\ 4a_{31} & 4\ a_{32} & 4\ a_{33} \end{vmatrix}$$

$$= 4 \times 8 \times 16 \times 2 \times 4 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 2^2 . 2^3 . 2^4 . 2^1 . 2^2 . 2^1 = 2^{13}$$

Hindi. Ans (D)

दिया गया है $P = [a_{ij}]_{3\times3}$ $\quad b_{ij} = 2^{i+j}$ aij

$Q = [b_{ij}]_{3\times3}$

$$P = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad |P| = 2 \quad ; \quad Q = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 4a_{11} & 8\ a_{12} & 16\ a_{13} \\ 8a_{21} & 16\ a_{22} & 32\ a_{23} \\ 16a_{31} & 32\ a_{32} & 64\ a_{33} \end{bmatrix}$$

Q का सारणिक $= \begin{vmatrix} 4a_{11} & 8\ a_{12} & 16\ a_{13} \\ 8a_{21} & 16\ a_{22} & 32\ a_{23} \\ 16a_{31} & 32\ a_{32} & 64\ a_{33} \end{vmatrix} = 4 \times 8 \times 16 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & 2\ a_{22} & 2\ a_{23} \\ 4a_{31} & 4\ a_{32} & 4\ a_{33} \end{vmatrix}$

$$= 4 \times 8 \times 16 \times 2 \times 4 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 2^2 . 2^3 . 2^4 . 2^1 . 2^2 . 2^1 = 2^{13}$$

13. If P is a 3×3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is the 3×3 identity matrix, then there exists a column matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ne \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that **[IIT-JEE 2012, Paper-2, (3, –1), 66]**

एक 3×3 आव्यूह (matrix) P इस प्रकार का है कि $P^T = 2P + I$, जहाँ P^T आव्यूह P का परिवर्त आव्यूह (transpose) और

I 3×3 का तत्समक आव्यूह है। तब एक स्तम्भ आव्यूह (column matrix) $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ne \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ का अस्तित्व इस प्रकार है कि

(A) $PX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (B) PX = X (C) PX = 2X (D*) PX = – X

Sol. **Ans. (D)**

$P^T = 2P + I \Rightarrow (P^T)^T = (2P + I)^T \Rightarrow P = 2P^T + I$

$\Rightarrow P = 2(2P + I) + I \Rightarrow 3P = -3I \Rightarrow P = -I$

$\Rightarrow PX = -IX = -X$

Hindi. $P^T = 2P + I \Rightarrow (P^T)^T = (2P + I)^T \Rightarrow P = 2P^T + I$

$\Rightarrow P = 2(2P + I) + I \Rightarrow 3P = -3I \Rightarrow P = -I$

$\Rightarrow PX = -IX = -X$

14*. If the adjoint of a 3 × 3 matrix P is $\begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$, then the possible value(s) of the determinant of P is (are)

[IIT-JEE 2012, Paper-2, (4, 0), 66]

यदि 3 × 3 आव्यूह (matrix) P का सहखंडज (adjoint) $\begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$ है तो P के सारणिक (determinant) का (के) सम्भावित मान है (हैं)

[IIT-JEE 2012, Paper-2, (4, 0), 66]

(A*) –2 (B) –1 (C) 1 (D*) 2

Sol. **Ans. (AD)**

Let $A = [a_{ij}]_{3\times3}$; adj $A = \begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$

$|\text{adj } A| = 1(3 - 7) - 4(6 - 7) + 4(2 - 1) = 4$

$\Rightarrow |A|^{3-1} = 4 \Rightarrow |A|^2 = 4 \Rightarrow |A| = \pm 2$

Hindi. माना $A = [a_{ij}]_{3\times3}$; adj $A = \begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$

$|\text{adj } A| = 1(3 - 7) - 4(6 - 7) + 4(2 - 1) = 4$

$\Rightarrow |A|^{3-1} = 4 \Rightarrow |A|^2 = 4 \Rightarrow |A| = \pm 2$

15.* For 3×3 matrices M and N, which of the following statement(s) is (are) **NOT** correct ?

[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]

(A) $N^T M N$ is symmetric or skew symmetric, according as M is symmetric or skew symmetric

(B) M N – N M is skew symmetric for all symmetric matrices M and N

(C*) M N is symetric for all symmetric matrices M and N

(D*) (adj M) (adj N) = adj(MN) for all invertible matrices M and N

3×3 आव्यूहों M तथा N के लिए निम्न में से कौन प्रकथन सत्य **नहीं** हैं (हैं) ?

[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]

(A) M के सममित या विषम–सममित होने के अनुसार $N^T M N$ सममित या विषम–सममित है।

(B) सभी सममित आव्यूहों M तथा N के लिए MN – NM विषम –सममित है।

(C*) सभी सममित आव्यूहों M तथा N के लिए MN सममित है।

(D*) सभी व्युत्क्रमणीय आव्यूहों M तथा N के लिए (adj M) (adj N) = adj(MN)

Sol. **(C,D)**

(A) $(N^TMN)^T = N^T M^T N$ is symmetric if M is symmetric and skew-symmetric if M is skew-symmetric.

(B) $(MN - NM)^T = (MN)^T - (NM)^T = NM - MN = -(MN - NM)$
skew symmetric

(C) $(MN)^T = N^T M^T = NM \neq MN$ hence NOT correct

(D) standard result is

$$adj(MN) = (adjN)\ (adj\ M)] \neq (adjM)\ (adjN)$$

Hindi. (C,D)

(A) $(N^TMN)^T = N^T M^T N$ सममित है यदि M सममित है और विषम सममित है यदि M विषम सममित है।

(B) $(MN - NM)^T = (MN)^T - (NM)^T = NM - MN = -(MN - NM)$
विषम सममित

(C) $(MN)^T = N^T M^T = NM \neq MN$ अतः सही नही है

(D) मानक परिणाम

$$adj(MN) = (adjN)\ (adj\ M)] \neq (adjM)\ (adjN)$$

16*. Let M be a 2 × 2 symmetric matrix with integer entries. Then M is invertible if
[JEE (Advanced) 2014, Paper-1, (3, 0)/60]

(A) the first column of M is the transpose of the second row of M
(B) the second row of M is the transpose of first column of M
(C*) M is a diagonal matrix with nonzero entries in the main diagonal
(D*) the product of entries in the main diagonal of M is not the square of an integer

माना कि 2 × 2 सममित आव्यूह (symmetric matrix) M के सभी अवयव (elements) पूर्णांक (integer) हैं। तब M व्युत्क्रमणीय (invertible) है, यदि

(A) M का पहला स्तम्भ M की दूसरी पक्ति का परिवर्त (transpose) है।
(B) M की दूसरी पंक्ति M के पहले स्तम्भ का परिवर्त है।
(C*) M एक विकर्ण आव्यूह (diagonal matrix) है जिसके मुख्य विकर्ण (main diagonal) के अवयव शून्यतर (non-zero) हैं
(D*) M के मुख्य विकर्ण (main diagonal) के अवयवों का गुणनफल किसी भी पूर्णांक का वर्ग नहीं है।

Ans. (CD)

Sol. $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

(A) $\begin{bmatrix} a \\ b \end{bmatrix}$ & [b c] are transpose.

So $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}$ is given $\Rightarrow$ $a = b = c$

$M = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$ $\Rightarrow$ $|M| = 0$ A is wrong.

(B) [b c] & $\begin{bmatrix} a \\ b \end{bmatrix}$ are transpose.

So a = b = c B is wrong

(C) $M = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$ $\Rightarrow$ $|M| = ac \neq 0$ C is correct

(D) $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ given $ac \neq \lambda^2$. D is correct

(C, D) are correct.

Hindi. $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

(A) $\begin{bmatrix} a \\ b \end{bmatrix}$ तथा [b c] परिवर्त आव्यूह

$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}$ = दिया है $\Rightarrow$ $a = b = c$

$M = \begin{bmatrix} a & a \\ a & a \end{bmatrix} \Rightarrow |M| = 0$ A गलत है

(B) $[b\ c]$ & $\begin{bmatrix} a \\ b \end{bmatrix}$ परिवर्त आव्यूह

$a = b = c$ B गलत है

(C) $M = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \Rightarrow |M| = ac \neq 0$ C सही है

(D) $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ given $ac \neq \lambda^2$ D सही है

(C, D) सही है।

17*. Let M and N be two 3×3 matrices such that MN = NM. Further, if $M \neq N^2$ and $M^2 = N^4$, then **[JEE (Advanced) 2014, Paper-1, (3, 0)/60]**

(A*) determinant of $(M^2 + MN^2)$ is 0
(B*) there is a 3×3 non-zero matrix U such that $(M^2 + MN^2)U$ is the zero matrix
(C) determinant of $(M^2 + MN^2) \geq 1$
(D) for a 3×3 matrix U, if $(M^2 + MN^2)U$ equals the zero matrix then U is the zero matrix

माना कि दो 3×3 आव्यूह (matrices) M तथा N इस प्रकार है कि MN = NM है। यदि $M \neq N^2$ तथा $M^2 = N^4$ हो, तो

(A*) $(M^2 + MN^2)$ के सारणिक (determinant) का मान शून्य है।
(B*) एक ऐसा 3×3 शून्येतर (non-zero) आव्यूह U है जिसके लिये $(M^2 + MN^2)U$ शून्य आव्यूह है।
(C) $(M^2 + MN^2)$ के सारणिक मान ≥ 1 है।
(D) 3×3 आव्यूह U जिसके लिये $(M^2 + MN^2)U$ शून्य आव्यूह है तो U भी एक शून्य आव्यूह होगा।

Ans. **(AB)**

Sol. MN = NM & $M^2 - N^4 = 0$

$(M - N^2)(M + N^2) = 0$

$M - N^2 = 0$ Not Possible
$M + N^2 = 0$, $M - N^2 \neq 0$
$|M + N^2| = 0$, $|M - N^2| = 0$

In any case $|M + N^2| = 0$

(A) $|M^2 + MN^2| = |M|\,|M + N^2|$
$= 0$
(A) is correct

(B) If $|A| = 0$ then AU = 0 will have ∞ solution.
Thus $(M^2 + MN^2)$ U = 0 will have many 'U'
(B) is correct

(C) Obvious wrong.

(D) If AX = 0 & $|A| = 0$ then X can be non zero.
(D) is wrong

Hindi. MN = NM तथा $M^2 - N^4 = 0$

$(M - N^2)(M + N^2) = 0$

$M - N^2 = 0$ Not Possible
$M + N^2 = 0$, $M - N^2 \neq 0$
$|M + N^2| = 0$, $|M - N^2| = 0$

In any case $|M + N^2| = 0$

(A) $|M^2 + MN^2| = |M|\,|M + N^2|$
$= 0$
(A) सही है

(B) यदि $|A| = 0$ तब $AU = 0$ अनन्त हल रखेगा

अतः $(M^2 + MN^2)\, U = 0$ इसप्रकार के अनन्त 'U' होंगें

(B) सही है

(C) स्पष्टतः गलत है।

(D) यदि $AX = 0$ और $|A| = 0$ तब X अशून्य हो सकता है।

(D) गलत है।

18*. Let X and Y be two arbitrary, 3 × 3, non-zero, skew-symmetric matrices and Z be an arbitrary 3 × 3, non-zero, symmetric matrix. Then which of the following matrices is (are) skew symmetric ?

माना कि X एवं Y दो स्वेच्छ (arbitrary), 3 × 3, शून्येतर (non-zero) विषम सममित (skew-symmetric) आव्यूह (Matrix) है और Z एक स्वेच्छ, 3 × 3, शून्येतर, सममित (symmetric) आव्यूह है। तब निम्नलिखित में से कौनसा (से) विषम सममित आव्यूह है (हैं) ? **[JEE (Advanced) 2015, Paper-1 (4, –2)/88]**

(A) $Y^3Z^4 - Z^4Y^3$ (B) $X^{44} + Y^{44}$ (C*) $X^4Z^3 - Z^3X^4$ (D*) $X^{23} + Y^{23}$

Ans. **(C,D)**

Sol. **(C)** $(x^4 Z^3 - Z^3 X^4)^T = (X^4Z^3)^T (Z^3 X^4)^T$

$= (Z^T)^3(X^T)^4 - (X^T)^4(Z^T)^3$

$= Z^3X^4 - X^4Z^3$

$= -(X^4Z^3 - Z^3X^4)$

(D) $(X^{23} + Y^{23})^T = -X^{23} - Y^{23}$ $\Rightarrow$ $X^{23} + Y^{23}$ is skew–symmetric विषम सम्मित है

19*. Which of the following values of α satisfy the equation $\begin{vmatrix}(1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2\\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2\\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2\end{vmatrix} = -648\alpha$?

α के निम्नलिखित मानों में कौन सा (से) मान समीकरण $\begin{vmatrix}(1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2\\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2\\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2\end{vmatrix} = -648\alpha$ को संतुष्ट करता (करतें) है (है।) ? **[JEE (Advanced) 2015, Paper-1 (4, –2)/88]**

(A) – 4 (B*) 9 (C*) – 9 (D) 4

Ans. **(B,C)**

Sol. $R_3 \to R_3 - R_2,\ R_2 \to R_2 - R_1$

$$\begin{vmatrix}(1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2\\ 3+2\alpha & 3+4\alpha & 3+6\alpha\\ 5+2\alpha & 5+4\alpha & 5+6\alpha\end{vmatrix} = -648\alpha$$

$R_3 \to R_3 - R_2$

$$\begin{vmatrix}(1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2\\ 3+2\alpha & 3+4\alpha & 3+6\alpha\\ 2 & 2 & 2\end{vmatrix} = -648\alpha$$

$C_3 \to C_3 - C_2,\ C_2 \to C_2 - C_1$

$$\begin{vmatrix}(1+\alpha)^2 & \alpha(2+3\alpha) & \alpha(2+5\alpha)\\ 3+2\alpha & 2\alpha & 2\alpha\\ 2 & 0 & 0\end{vmatrix} = -648\alpha$$

$\Rightarrow$ $2\alpha^2(2+3\alpha) - 2\alpha^2(2+5\alpha) = -324\alpha$

$\Rightarrow$ $-4\alpha^3 = -324\alpha$ $\Rightarrow$ $\alpha = 0, \pm 9$

20. Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, where $\alpha \in R$. Suppose $Q = [q_{ij}]$ is a matrix such that $PQ = k\,I$, where $k \in R$, $k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$ and $\det(Q) = \frac{k^2}{2}$, then

[JEE (Advanced) 2016, Paper-1 (4, –2)/62]

माना कि $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, जहाँ $\alpha \in R$ है। मान लीजिए कि $Q = [q_{ij}]$ एक ऐसा आव्यूह (matrix) है कि $PQ = k\,I$, जहाँ $k \in R$, $k \neq 0$ और I तीन कोटि (order 3) का तत्समक आव्यूह (identity matrix) है। यदि $q_{23} = -\frac{k}{8}$ और det $(Q) = \frac{k^2}{2}$ हो , तब

(A) $\alpha = 0, k = 8$ (B) $4\alpha - k + 8 = 0$ (C) $\det(P \operatorname{adj}(Q)) = 2^9$ (D) $\det(Q \operatorname{adj}(P)) = 2^{13}$

Ans. **(B,C)**

Sol. As चूंकि $PQ = kI \Rightarrow Q = kP^{-1}I$

now अब $Q = \frac{k}{|P|}(\operatorname{adj}P)\,I \Rightarrow Q = \frac{k}{(20+12\alpha)}\begin{bmatrix} - & - & - \\ - & - & (-3\alpha-4) \\ - & - & - \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\because q_{23} = \frac{-k}{8} \Rightarrow \frac{k}{(20+12\alpha)}(-3\alpha-4) = \frac{-k}{8} \Rightarrow 2(3\alpha+4) = 5 + 3\alpha$

$3\alpha = -3 \Rightarrow \alpha = -1$

also और भी $|Q| = \frac{k^3\,|I|}{|P|} \Rightarrow \frac{k^2}{2} = \frac{k^3}{(20+12\alpha)}$

$(20 + 12\alpha) = 2k \Rightarrow 8 = 2k \Rightarrow k = 4$

(A) incorrect गलत

(B) correct सही

(C) $|P(\operatorname{adj}Q)| = |P|\,|\operatorname{adj}Q| = P|\,|Q|^2 = 2^2(2^3)^2 = 2^9$ correct सही

(D) $|Q(\operatorname{adj}P)| = |Q|\,|\operatorname{adj}P| = |Q|\,|P|^2 = 2^3(2^3)^2 = 2^9$ incorrect गलत

21. The total number of distinct $x \in R$ for which $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$ is

ऐसे सभी भिन्न (distinct) $x \in R$ जिनके लिए $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$ है, की कुल संख्या है–

Ans. 2 **[JEE (Advanced) 2016, Paper-1 (3, 0)/62]**

Sol. $x.x^2 \begin{vmatrix} 1 & 1 & 1+x^3 \\ 0 & 2 & 6x^3-1 \\ 0 & 6 & 24x^3-2 \end{vmatrix} = 10 \quad \Rightarrow \quad x^3(12x^3 + 2) = 10$

$6x^6 + x^3 - 5 = 0 \quad \Rightarrow \quad 6x^6 + 6x^3 - 5x^3 - 5 = 0$

$(6x^3 - 5)(x^3 + 1) = 0 \quad \Rightarrow \quad x^3 = -1,\ x^3 = \dfrac{5}{6}$

$x = -1,\ x = \left(\dfrac{5}{6}\right)^{1/3}$ so two solutions अत: दो हल = 2

22. Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3. If $Q = [q_{ij}]$ is a matrix such that $P^{50} - Q = I$, then $\dfrac{q_{31}+q_{32}}{q_{21}}$ equals **[JEE (Advanced) 2016, Paper-2 (3, –1)/62]**

माना कि $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ और I तीन कोटि (order 3) का तत्समक आव्यूह (identity matrix) है। यदि $Q = [q_{ij}]$ एक आव्यूह इस प्रकार है कि $P^{50} - Q = I$ है, तब $\dfrac{q_{31}+q_{32}}{q_{21}}$ का मान है–

(A) 52 (B) 103 (C) 201 (D) 205

Ans. **(B)**

Sol. $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix} \Rightarrow P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 16(1+2) & 8 & 1 \end{bmatrix},\ P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ 16(1+2+3) & 12 & 1 \end{bmatrix}$

$P^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 4\times 50 & 1 & 0 \\ 16(1+2+\ldots+50) & 4\times 50 & 1 \end{bmatrix} \quad \frac{\theta}{2} \quad P^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 200 & 1 & 0 \\ 20400 & 200 & 1 \end{bmatrix} \quad \because P^{50} - Q = I$

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 200 & 1 & 0 \\ 20400 & 200 & 1 \end{bmatrix} - \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow 200 - q_{21} = 0 \Rightarrow q_{21} = 200$

$20400 - q_{31} = 0 \Rightarrow q_{31} = 20400$ and और $200 - q_{32} = 0 \Rightarrow q_{32} = 200$

$\therefore \dfrac{q_{31}+q_{32}}{q_{21}} = \dfrac{20400+200}{200} = 103$

23. Let a, λ, $\mu \in$ R. Consider the system of linear equations

$$ax + 2y = \lambda$$
$$3x - 2y = \mu$$

[JEE (Advanced) 2016, Paper-2 (4, –2)/62]

Which of the following statement(s) is(are) correct ?

(A) if a = – 3, then the system has infinitely many solutions for all values of λ and μ

(B) If a $\neq$ –3, then the system has a a unique solution for all values of λ and μ

(C) If $\lambda + \mu = 0$, the the system has infinitely many solutions for a = –3

(D) If $\lambda + \mu \neq 0$, then the system has no solution for a = –3

माना कि a, λ, $\mu \in$ R हैं। इन रैखिक समीकरणों के निकाय (system of linear equations) पर विचार कीजिए

$$ax + 2y = \lambda$$
$$3x - 2y = \mu$$

निम्नलिखित में से कौन सा (से) कथन सही है (हैं) ?

(A) यदि a = – 3, तब λ तथा μ के सभी मानों के लिए निकाय के अनन्त (infinitely many) हल हैं।

(B) यदि a $\neq$ –3, तब λ तथा μ के सभी मानों के लिए निकाय का अद्वितीय (unique) हल हैं।

(C) यदि $\lambda + \mu = 0$, तब a = –3 के लिए निकाय के अनन्त हल हैं।

(D) यदि $\lambda + \mu \neq 0$, तब a = –3 के लिए निकाय का कोई हल नहीं हैं।

Ans. (B,C,D)

Sol. $ax + 2y = \lambda$

$3x - 2y = \mu$

(A) a = – 3 gives दिया गया है

$\lambda = -\mu$

or या $\lambda + \mu = 0$ not for all λ, μ (सभी λ, μ के लिए नहीं है)

(B) $a \neq -3 \Rightarrow \Delta \neq 0$ where जहाँ $\Delta = \begin{vmatrix} a & 2 \\ 3 & -2 \end{vmatrix} = -2a - 6$

$\therefore$ (B) is correct सही है

(C) correct सही

(D) if यदि $\lambda + \mu \neq 0$

$\Rightarrow$ $-3x + 2y = \lambda$ ....(1)

& तथा $3x - 2y = \mu$ ....(2)

inconsistent असंगत $\Rightarrow$ (D) correct सही

24. Which of the following is(are) NOT the square of a 3×3 matrix with real entries?

[JEE(Advanced) 2017, Paper-1,(4, –2)/61]

निम्न में से कौनसा(से) वास्तविक संख्याओं के 3×3 आव्यूह (matrix) का वर्ग (square) नहीं है(हैं)?

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (C) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ans. **(A,C)**

Sol. $A = B^2 \Rightarrow |A| = |B|^2 = +ve$

(A) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 1(-1) = \text{negative}$ (ऋणात्मक)

Matrix B can not be possible (आव्यूह B सम्भव नहीं हैं)

(B) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 1\ (1-0) = \text{positive}$ (धनात्मक)

Matrix B can be possible (आव्यूह B सम्भव हैं)

Ex. $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$

(C) $\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -1 = \text{negative}$ (ऋणात्मक)

Matrix B can not be possible (आव्यूह B सम्भव नहीं हैं)

(D) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 = \text{positive}$ (धनात्मक)

Matrix B can be I (आव्यूह B, I सम्भव हैं)

25. For a real number α, if the system **[JEE(Advanced) 2017, Paper-1,(3, 0)/61]**

$$\begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

of linear equations, has infinitely many solutions, then $1 + \alpha + \alpha^2 =$

वास्तविक संख्या (real number) α के लिए, यदि रैखिक समीकरण निकाय (system of linear equations)

$$\begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

के अनन्त हल (infinitely many solutions) हैं, तब $1 + \alpha + \alpha^2 =$

Ans. **(1)**

Sol. $D = 0$

$$\begin{vmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} (1+\alpha+\alpha^2) & (2\alpha+1) & (\alpha^2+\alpha+1) \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} (1+\alpha+\alpha^2) & (2\alpha+1) & 0 \\ \alpha & 1 & 0 \\ \alpha^2 & \alpha & 1-\alpha^2 \end{vmatrix} = 0 \Rightarrow (1-\alpha^2)(1+\alpha+\alpha^2-2\alpha^2-\alpha)=0 \quad \Rightarrow (1-\alpha^2)=0$$

$\alpha = -1$ or 1

for $\alpha = 1$, system of linear equations has no solution

$\alpha = 1$ के लिए सरल रेखा निकाय का कोई हल नहीं है।

$\therefore \quad \alpha = -1$ so अत: $1 + \alpha + \alpha^2 = 1$

26. How many 3 × 3 matrices M with entries from {0, 1, 2} are there, for which the sum of the diagonal entries of M^TM is 5 ? **[JEE(Advanced) 2017, Paper-2,(3, –1)/61]**

ऐसे कितने 3 × 3 आव्यूह M हैं जिनकी प्रविष्टियाँ (entries) {0, 1, 2} में हैं एवम् $M^T M$ की विकर्णीय प्रविष्टियों (diagonal elements) का योग 5 है?

(A) 198 (B) 162 (C) 126 (D) 135

Ans. **(A)**

Sol. $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$

$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2 + i^2 = 5$

Case स्थिति -I : Five पाँच (1's) and four और चार (0's)

$^9C_5 = 126$

Case- स्थिति II : One एक (2) and one और एक (1)

$^9C_2 \times 2! = 72 \quad \therefore$ Total कुल = 198

27. Let S be the set of all column matrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1, b_2, b_2 \in R$ and the system of equation (in real variables)

$$-x + 2y + 5z = b_1$$
$$2x - 4y + 3z = b_2$$
$$x - 2y + 2z = b_3$$

has at least one solution. Then, which of the following system(s) (in real variables) has (have) at least one solution for each $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$? **[JEE(Advanced) 2018, Paper-2,(4, –2)/60]**

(A) $x + 2y + 3z = b_1$, $4y + 5z = b_2$ and $x + 2y + 6z = b_3$
(B) $x + y + 3z = b_1$, $5x + 2y + 6z = b_2$ and $-2x - y - 3z = b_3$
(C) $-x + 2y - 5z = b_1$, $2x - 4y + 10z = b_2$ and $x - 2y + 5z = b_3$
(D) $x + 2y + 5z = b_1$, $2x + 3z = b_2$ and $x + 4y - 5z = b_3$

माना कि S उन सभी स्तम्भ आव्यूहो (column matrices) $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ का समुच्चय (set) है जिनके लिए $b_1, b_2, b_2 \in R$ और वास्तविक चरों (real variables) वालें समीकरण निकाय (system of equations)

$$-x + 2y + 5z = b_1$$
$$2x - 4y + 3z = b_2$$
$$x - 2y + 2z = b_3$$

का कम से कम एक हल (solution) है। तब निम्नलिखित वास्तविक चरों वाले निकायों में से किस (कौन से) निकाय (निकायों) का भी प्रत्येक $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$ के लिए कम से कम एक हल है ?

(A) $x + 2y + 3z = b_1$, $4y + 5z = b_2$ और $x + 2y + 6z = b_3$

(B) $x + y + 3z = b_1$, $5x + 2y + 6z = b_2$ और $-2x - y - 3z = b_3$

(C) $-x + 2y - 5z = b_1$, $2x - 4y + 10z = b_2$ और $x - 2y + 5z = b_3$

(D) $x + 2y + 5z = b_1$, $2x + 3z = b_2$ और $x + 4y - 5z = b_3$

Ans. **(AD)**

Sol. $\Delta = 0$ so for at least one solutions $\Delta_1 = \Delta_2 = \Delta_3 = 0 \Rightarrow b_1 + 7b_2 = 13b_3$ ....(1)

$\Delta = 0$ इसलिए कम से कम एक हल के लिए $\Delta_1 = \Delta_2 = \Delta_3 = 0 \Rightarrow b_1 + 7b_2 = 13b_3$ ....(1)

option (A) $\Delta \neq 0$ $\Rightarrow$ unique solution $\Rightarrow$ option (A) is correct

विकल्प (A) के लिए $\Delta \neq 0$ $\Rightarrow$ अद्वितीय हल $\Rightarrow$ विकल्प (A) सही है।

option (D) $\Delta \neq 0$ $\Rightarrow$ unique solution $\Rightarrow$ option (D) is correct

विकल्प (D) के लिए $\Delta \neq 0$ $\Rightarrow$ अद्वितीय हल $\Rightarrow$ विकल्प (D) सही है।

option (C) $\Delta = 0$ $\Rightarrow$ equations are $x - 2y + 5z = -b_1$

विकल्प (C) के लिए $\Delta = 0$ $\Rightarrow$ समीकरण $x - 2y + 5z = -b_1$

$$x - 2y + 5z = \frac{b_2}{2}$$

$$x - 2y + 5z = b_2$$

There planes are parallel so they must be coincident

तीनों समतल समान्तर है इसलिए वे समपाती होगे।

$\Rightarrow -b_1 = \frac{b_2}{2} = b_3$

All b_1, b_2, b_3 obtained from equation (1) may not satisfy this relation so option (C) is wrong.

समीकरण (1) से प्राप्त सभी b_1, b_2, b_3 इस समीकरण को संतुष्ट नहीं करते है। अतः विकल्प (C) गलत है।

option विकल्प (B) के लिए $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 5 & 2 & 2 \\ 2 & 1 & 1 \end{vmatrix} = 0$. Also तथा $\Delta_1 = 0$

For infinite solutions, Δ_2 and Δ_3 must be 0

अनन्त हल के लिए Δ_2 और Δ_3, 0 होगा।

$\Rightarrow \begin{vmatrix} 1 & b_1 & 1 \\ 5 & b_2 & 2 \\ 2 & b_3 & 1 \end{vmatrix} = 0 \Rightarrow -b_1 - b_2 + 3b_3 = 0$ which does not satisfy (1) for all b_1, b_2, b_3 so option(B) wrong

$\Rightarrow \begin{vmatrix} 1 & b_1 & 1 \\ 5 & b_2 & 2 \\ 2 & b_3 & 1 \end{vmatrix} = 0 \Rightarrow -b_1 - b_2 + 3b_3 = 0$ जो कि समीकरण (1) को संतुष्ट नहीं करता है सभी b_1, b_2, b_3 इसलिए विकल्प (B) गलत है।

28. Let P be a matrix of order 3×3 such that all the entries in P are from the set $\{-1, 0, 1\}$. Then, the maximum possible value of the determinant of P is _____ . **[JEE(Advanced) 2018, Paper-2,(3, 0)/60]**

माना कि P, 3×3 कोटि (order) का एक ऐसा आव्यूह (matrix) है कि P की सभी प्रविशिष्टयॉ समुच्चय (set) $\{-1, 0, 1\}$ में से है। P के सारणिक (determinant) का अधिकतम् संभावित मान (maximum possible value) है _____।

Ans. **(4)**

Sol. det (P) = $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ = $a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \le 6$

value can be 6 only if $a_1 = 1, a_2 = -1, a_3 = 1, b_2c_3 = b_1c_3 = b_1c_2 = 1, b_3c_2 = b_3c_1 = b_2c_1 = -1$

मान केवल 6 हो सकते है यदि $a_1 = 1, a_2 = -1, a_3 = 1, b_2c_3 = b_1c_3 = b_1c_2 = 1, b_3c_2 = b_3c_1 = b_2c_1 = -1$

$\Rightarrow$ $(b_2c_3)(b_3c_1)(b_1c_2) = -1$ & और $(b_1c_3)(b_3c_2)(b_2c_1) = 1$

i.e. $b_1b_2b_3c_1c_2c_3 = 1$ and और -1

hence not possible अतः संभव नहीं

Similar contradiction occurs when $a_1 = 1, a_2 = 1, a_3 = 1, b_2c_2 = b_3c_1 = b_1c_2 = 1$ $b_3c_2 = b_1c_3 = b_1c_2 = -1$

इसी प्रकार विरोधाभास आता है जब $a_1 = 1, a_2 = 1, a_3 = 1, b_2c_2 = b_3c_1 = b_1c_2 = 1$ $b_3c_2 = b_1c_3 = b_1c_2 = -1$

Now for value to be 5 one the terms must be zero but that will make 2 terms zero which means answer cannot be 5

अब 5 के लिए एक पद अवश्य शून्य होगा परन्तु यहाँ 2 पद शून्य बनते है अतः 5 उत्तर नहीं हो सकता

Now $\begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{vmatrix}$ = 4 Hence max value = 4

अब $\begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{vmatrix}$ = 4 अतः अधिकतम मान = 4

29. Let M = $\begin{bmatrix} \sin^4\theta & -1-\sin^2\theta \\ 1+\cos^2\theta & \cos^4\theta \end{bmatrix} = \alpha I + \beta M^{-1}$ **[Matrices & Determinants] [T]**

Where $\alpha = \alpha(\theta)$ and $\beta = \beta(\theta)$ are real numbers and I is the 2 × 2 identity matrix.

If α^* = is the minimum of the set $\{a(\theta) : \theta \in [0, 2\pi]\}$ and **{[MT-AI]-T-305}**

β^* = is the minimum of the set $\{\beta(\theta) : \theta \in [0, 2\pi)\}$ **[JEE(Advanced) 2019, Paper-1,(4, –1)/62]**

Then the value of $\alpha^* + \beta^*$ is

माना कि M = $\begin{bmatrix} \sin^4\theta & -1-\sin^2\theta \\ 1+\cos^2\theta & \cos^4\theta \end{bmatrix} = \alpha I + \beta M^{-1}$

जहाँ $\alpha = \alpha(\theta)$ और $\beta = \beta(\theta)$ वास्तविक (real) संख्याएँ है, और I एक 2 × 2 तत्समक-आव्यूह (2 × 2 identity matrix) है। यदि

समुच्चय $\{a(\theta) : \theta \in [0, 2\pi]\}$ का निम्नतम (minimum) α^* है और

समुच्चय $\{\beta(\theta) : \theta \in [0, 2\pi)\}$ का निम्नतम (minimum) β^* है,

तो $\alpha^* + \beta^*$ का मान है -

(A) $\frac{-37}{16}$ (B) $\frac{-29}{16}$ (C) $\frac{-31}{16}$ (D) $\frac{-17}{16}$

Ans. **(B)**

Sol. m = $\sin^4\theta \cdot \cos^4\theta + (1 + \sin^2\theta)(1 + \cos^2\theta)$

$2 + \sin^4\cos^4\theta + \sin^2\theta\cos^2\theta$

$$\begin{bmatrix} \sin^4\theta & -(1+\sin^2\theta) \\ 1+\cos^2\theta & \cos^4\theta \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} + \beta = \frac{1}{|m|}\begin{bmatrix} \cos^4\theta & 1+\sin^2\theta \\ -1-\cos^2\theta & \sin^4\theta \end{bmatrix}$$

$\sin^4\theta = \dfrac{\alpha+\beta}{|m|}\cos^4\theta$, $-1-\sin^2\theta = \dfrac{\beta}{|m|}(1+\sin^2\theta)$

$\beta = -|m|$

$\beta = -[\sin^4\theta\cos^4\theta + \sin^2\theta\cos^2\theta + 2] = -[t^2 + t + 2] \Rightarrow \quad \beta_{min} = -\dfrac{37}{16}$

$\alpha = \sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta\cos^2\theta = 1 - \dfrac{1}{2}(\sin^2 2\theta) \quad \Rightarrow \quad \min\alpha = \dfrac{1}{2}$

$\alpha + \beta = -\dfrac{37}{16} + \dfrac{1}{2} = -\dfrac{37}{16} + \dfrac{8}{16} = -\dfrac{29}{16}$

30. Let $M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix}$ and adj $M = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$ **{Matrices Determinant[MT-AI]-M-305}**

where a and b are real numbers. Which of the following options is/are correct ?

(A) $\det(\text{adj } M^2) = 81$

(B) $a + b = 3$

(C) If $M\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then $\alpha - \beta + \gamma = 3$

(D) $(\text{adj } M)^{-1} + \text{adj } M^{-1} = -M$

[JEE(Advanced) 2019, Paper-1,(4, –1)/62]

माना कि $M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix}$ और adj $M = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

जहाँ a और b वास्तविक संख्याएँ (real numbers) है। निम्न में से कौन सा (से) विकल्प सही है (है) ?

(A) $\det(\text{adj } M^2) = 81$

(B) $a + b = 3$

(C) यदि $M\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, तब $\alpha - \beta + \gamma = 3$

(D) $(\text{adj } M)^{-1} + \text{adj } M^{-1} = -M$

Ans. (BCD)

Sol. $M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix}$ and और adj $M = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

$(\text{adj}M)_{11} = 2 - 3b$, $(\text{adj}M)_{22} = -3a$

$\Rightarrow 2 - 3b = -1$

$\Rightarrow b = 1$ & $-3a = -6 \Rightarrow a = 2$

$|\text{adj}M| = -1(6-6) - 1(-8+10) - 1(24-30) = 4$

$\det\{\text{adj}(M^2)\} = |\det(\text{adj}M)|^2 = 16$

Now अब $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \beta + 2\gamma = 1,\ \alpha + 2\beta + 3\gamma = 2,\ 3\alpha + \beta + \gamma = 3$

On solving $\alpha = 1, \beta = -1, \gamma = 1$ so $\alpha - \beta + \gamma = 3$

Now $(\text{adj } M)^{-1} + (\text{adj } M)^{-1} = 2(\text{adj } M)^{-1} = \dfrac{2\text{adj}(\text{adj}M)}{|\text{adj} M|} = \dfrac{1}{2}|M|^{3-2}M = -M$

31. Let $x \in R$ and let

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix},\ Q = \begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 6 \end{bmatrix} \text{ and } R = PQP^{-1}.$$

Then which of the following is/are correct

(A) there exists a real number x such that PQ= QP

(B) $\det R = \det \begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{bmatrix} + 8$ for all $x \in R$ **[Matrices & Determinants_T]** **{[MT-AI]-E-305}**

(C) For x = 1 there exists a unit vector $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ for which are $R\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(D) For x = 0 if $R\begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = 6\begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$ then a + b = 5

[JEE(Advanced) 2019, Paper-2 ,(4, –1)/62]

माना कि $x \in R$ और मानाकि

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix},\ Q = \begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 6 \end{bmatrix}$$ तथा $R = PQP^{-1}$ तो निम्न में से कौन सा (से) विकल्प सही है (हैं)–

(A) एक ऐसी वास्तविक संख्या x सम्भव है जिसके लिए PQ= QP

(B) सभी $x \in R$ के लिए $\det R = \det \begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{bmatrix} + 8$

(C) x = 1 के लिए, एक इकाई सदिश $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ सम्भव है, जिसके लिए $R\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(D) x = 0 के लिए यदि $R\begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = 6\begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$, तो a + b = 5

[JEE(Advanced) 2019, Paper-2 ,(4, –1)/62]

Ans. **(BD)**

Sol. $\det R = \det P \times \det Q \times \det (P^{-1})$

$\Rightarrow \quad \det R = \det Q = 4(12 - x^2) = 48 - 4x^2$

Now $\begin{vmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{vmatrix} = 4(10 - x^2) = 40 - 4x^2$

$\Rightarrow$ det R = det $\begin{pmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{pmatrix} + 8 \ \forall \ x \in R$

At x = 1, det Q = 48 – 4 = 44 = det R

Because det R $\neq$ 0 so

$R \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \alpha = \beta = \gamma = 0$

So $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ is not a unit vector

$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, $Q(x = 0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

$R = P\,Q\,P^{-1}$

$= \begin{bmatrix} 2 & 4 & 6 \\ 0 & 8 & 12 \\ 0 & 0 & 18 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 6 & -3 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 12 & 6 & 4 \\ 0 & 24 & 8 \\ 0 & 0 & 36 \end{bmatrix}$

$R = \begin{bmatrix} 2 & 1 & 2/3 \\ 0 & 4 & 4/3 \\ 0 & 0 & 6 \end{bmatrix}$

$(R - 6I) \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = \begin{bmatrix} -4 & 1 & 2/3 \\ 0 & -2 & 4/3 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}$

$-4 + a + \frac{2b}{3} = 0$

$-2a + \frac{4}{3}b = 0$

$-8 + \frac{8}{3} = 0 \Rightarrow b = 3$

a = 2

PQ = QP $\Rightarrow$ a_{12} for both are same

$\Rightarrow$ x + 4 + x = 2 + 2x + 0 $\Rightarrow$ $x \in \phi$ $\Rightarrow$ No value exist

Hindi. det R = det P × det Q × det (P^{-1})

$\Rightarrow$ det R = det Q = $4(12 - x^2) = 48 - 4x^2$

अब $\begin{vmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{vmatrix} = 4(10 - x^2) = 40 - 4x^2$

$\Rightarrow$ det R = det $\begin{pmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{pmatrix} + 8 \ \forall \ x \in R$

At x = 1 पर, det Q = 48 – 4 = 44 = det R

क्योंकि det R $\neq$ 0 इसलिए

$R \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \alpha = \beta = \gamma = 0$

इसलिए $\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$ एक इकाई सदिश नही है।

$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, $Q(x = 0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

$R = P\,Q\,P^{-1}$

$= \begin{bmatrix} 2 & 4 & 6 \\ 0 & 8 & 12 \\ 0 & 0 & 18 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 6 & -3 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 12 & 6 & 4 \\ 0 & 24 & 8 \\ 0 & 0 & 36 \end{bmatrix}$

$R = \begin{bmatrix} 2 & 1 & 2/3 \\ 0 & 4 & 4/3 \\ 0 & 0 & 6 \end{bmatrix}$

$(R - 6I) \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = \begin{bmatrix} -4 & 1 & 2/3 \\ 0 & -2 & 4/3 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}$

$-4 + a + \frac{2b}{3} = 0$

$-2a + \frac{4}{3}b = 0$

$-8 + \frac{8}{3} = 0 \quad \Rightarrow \quad b = 3$

$a = 2$

$PQ = QP \quad \Rightarrow \quad a_{12}$ दोनों के लिये समान है।

$\Rightarrow \quad x + 4 + x = 2 + 2x + 0 \quad \Rightarrow \quad x \in \phi \quad \Rightarrow$ कोई मान विद्यमान नहीं

32. Let $P_1 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$P_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $P_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $P_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$,

and $X = \sum_{k=1}^{6} P_k \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix} P_k^T$.

Where P_k^T denotes the transpose of matrix P_k. Then which of the following options is/are correct ?

[Matrices & Determinants_T] **{[MT-AI]-T-305}**

(A) X is a symmetric matrix

(B) The sum of diagonal entries of X is 18.

(C) If $X \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then $\alpha = 30$

(D) $X - 30I$ is an invertible matrix

[JEE(Advanced) 2019, Paper-2 ,(4, –1)/62]

माना कि

$P_1 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$$P_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad P_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad P_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

और $\quad X = \sum_{k=1}^{6} P_k \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix} P_k^T$

जहाँ आव्यूह (matrix) P_k के परिवर्त (transpose) को P_k^T से दर्शाया गया है। तब निम्न में से कौनसा (से) विकल्प सही है (हैं)–

(A) X एक सममित आव्यूह है।

(B) X के विकर्णो के अवयवों का योग 18 होगा।

(C) यदि $X \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, तो $\alpha = 30$

(D) $X - 30I$ एक व्युत्क्रमणीय आव्यूह होगा।

[JEE(Advanced) 2019, Paper-2 ,(4, –1)/62]

Ans. **(ABC)**

Sol. Clearly स्पष्टतया $P_1 = P_1^T = P_1^{-1}$

$P_2 = P_2^T = P_2^{-1}$

.

.

.

$P_6 = P_6^T = P_6^{-1}$

and और $A^T = A$, where जहां $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

Using formula सूत्र से $(A + B)^T = A^T + B^T$

$X^T = \left(P_1AP_1^T + ... + P_6AP_6^T\right)^T = P_1A^TP_1^T + ... + P_6A^TP_6^T = X \qquad \Rightarrow$ X is symmetric. सममित है।

Let माना $B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$XB = P_1AP_1^TB + P_2AP_2^TB + ... + P_6AP_6^TB \quad = P_1AB + PAB + ...+P_6AB$

$XB = (P_1 + P_2 + ...+P_6) \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$

$= \begin{bmatrix} 6\times2+3\times2+6\times2 \\ 6\times2+3\times2+6\times2 \\ 6\times2+3\times2+6\times2 \end{bmatrix} = \begin{bmatrix} 30 \\ 30 \\ 30 \end{bmatrix} = 30B \qquad \Rightarrow \alpha = 30$

since क्योंकि $X\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 30\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow \qquad (X - 30I)B = 0$ has a non trivial solution $B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow \qquad |X - 30I| = 0$

$X = P_1AP_1^T + ...+ P_6AP_6^T$

trace (x) = $t_r(P_1AP_1^T)$ + ……..+$t_r(P_6AP_6^T)$ = (2 + 0 + 1) + ….+(2 + 0 + 1) = 3 × 6 = 18

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

भाग - II : JEE (MAIN) / AIEEE (पिछले वर्षो) के प्रश्न

1. The number of 3 × 3 non-singular matrices, with four entries as 1 and all other entries as 0, is
(1) 5 (2) 6 (3*) at least 7 (4) less than 4
[AIEEE 2010 (8, –2), 144]

कोटि 3 × 3 वाले व्युत्क्रमणीय आव्यूहों, जिसमें चार प्रविष्टियाँ 1 है तथा शेष सभी 0 है, की कुल संख्या है–
(1) 5 (2) 6 (3*) कम–से–कम 7 (4) 4 से कम
[AIEEE 2010 (8, –2), 144]

Ans. **(3)**

Sol. Let A = $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ ⇒ det (A) = $a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)$

$= a_1b_2c_3 - a_1c_2b_3 + a_2c_1b_3 - a_2b_1c_3 + a_3b_1c_2 - a_3c_1b_2$

if any of the terms is non-zero, then det (A) will be non-zero and all the element of that term will be unity
Now there are 6 elements remaining out of which any one can be unity.
Hence number of non-singular matrices = $\underbrace{^6C_1}_{\text{choosing any one triplet}} \times \underbrace{^6C_1}_{\text{choosing any one element}}$

Hence correct option is (3)

Hindi माना A = $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ ⇒ det (A) = $a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)$

$= a_1b_2c_3 - a_1c_2b_3 + a_2c_1b_3 - a_2b_1c_3 + a_3b_1c_2 - a_3c_1b_2$

यदि कोई पद अशून्य है तब det (A) अशून्य होगा और उस पद के समस्त अवयव इकाई होंगे। अब 6 अवयव शेष होगें जिनमें से कोई भी एक इकाई हो सकता है। अतः व्युत्क्रमणीय आव्यूहों की संख्या = $\underbrace{^6C_1}_{\text{choosing any one triplet}}$ × $\underbrace{^6C_1}_{\text{choosing any one element}}$

अतः सही विकल्प (3) है।

2. Let A be a 2 × 2 matrix with non-zero entries and let $A^2 = I$, where I is 2 × 2 identity matrix.
Tr(A) = sum of diagonal elements of A and |A| = determinant of matrix A. **[AIEEE 2010 (4, –1), 144]**
Statement -1 : Tr(A) = 0
Statement -2 : |A| = 1
(1) Statement -1 is true, Statement-2 is true ; Statement -2 is not a correct explanation for Statement -1.
(2*) Statement-1 is true, Statement-2 is false.
(3) Statement -1 is false, Statement -2 is true.
(4) Statement -1 is true, Statement -2 is true; Statement-2 is a correct explanation for Statement-1.

माना A, 2 × 2 कोटि का अशून्य प्रविष्टियों वाला एक आव्यूह है और माना $A^2 = I$, जहाँ I एक 2 × 2 कोटि का तत्समक आव्यूह है। Tr(A) = आव्यूह A के विकर्ण पर स्थित प्रविष्टियों का योगफल है तथा |A| = आव्यूह A का सारणिक है।
[AIEEE 2010 (4, –1), 144]

प्रकथन-1 : Tr(A) = 0
प्रकथन-2 : |A| = 1
(1) प्रकथन-1 सत्य है, प्रकथन-2 सत्य है ; प्रकथन-2, प्रकथन-1 की सही व्याख्या नहीं है।

(2*) प्रकथन-1 सत्य है, प्रकथन-2 मिथ्या है।
(3) प्रकथन-1 मिथ्या है, प्रकथन-2 सत्य है।
(4) प्रकथन-1 सत्य है, प्रकथन-2 सत्य है ; प्रकथन-2, प्रकथन-1 की सही व्याख्या है।

Ans. **(2)**

Sol. Let A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$; $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow$ a + d = 0 and $a^2 + bc = 1$
$\Rightarrow$ Tr(A) = 0
Statement-1 is true
Statement-2 |A| = ad – bc = – a^2 – bc = – 1
Statement-1 is true statement-2 is false.
Hence correct option is (2)

Hindi माना A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$; $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow$ a + d = 0 और $a^2 + bc = 1$
$\Rightarrow$ Tr(A) = 0
कथन-1 सत्य है।
कथन-2 |A| = ad – bc = – a^2 – bc = – 1
कथन-1 सत्य है कथन -2 असत्य है।
अतः सही विकल्प (2) है।

3. Consider the system of linear equations : **[AIEEE 2010 (4, –1), 144]**

$x_1 + 2x_2 + x_3 = 3$
$2x_1 + 3x_2 + x_3 = 3$
$3x_1 + 5x_2 + 2x_3 = 1$

The system has
(1) exactly 3 solutions (2) a unique solution (3*) no solution (4) infinite number of solutions

निम्न रैखिक समीकरण निकाय को लीजिए : **[AIEEE 2010 (4, –1), 144]**

$x_1 + 2x_2 + x_3 = 3$
$2x_1 + 3x_2 + x_3 = 3$
$3x_1 + 5x_2 + 2x_3 = 1$

निकाय के
(1) केवल 3 हल है। (2) एकमात्र हल है । (3*) कोई हल नहीं है। (4) अनन्त हल है।

Ans. **(3)**

Sol. Equation (2) – equation (1) $\Rightarrow x_1 + x_2 = 0$
(3) – 2(1) $\Rightarrow x_1 + x_2 = -5$
No solution कोई हल नहीं
Hence correct option is (3)
अतः सही विकल्प (3) है।

4. Let A and B be two symmetric matrices of order 3. **[AIEEE 2011, I, (4, –1), 120]**
Statement-1 : A(BA) and (AB)A are symmetric matrices.
Statement-2 : AB is symmetric matrix if matrix multiplication of A with B is commutative.
(1) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(2*) Statement-1 is true, Statement-2 is true; Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1.
(3) Statement-1 is true, Statement-2 is false.
(4) Statement-1 is false, Statement-2 is true.

माना A और B कोटि 3 के दो सममित आव्यूह है– **[AIEEE 2011, I, (4, –1), 120]**

कथन-1 : A(BA) तथा (AB)A सममित आव्यूह हैं।

कथन-2 : AB एक सममित आव्यूह है यदि आव्यूहों A तथा B की गुणा क्रम–विनिमेयकारी है।

(1) कथन-1 सत्य है, कथन-2 सत्य है। कथन-2, कथन-1 की सही व्याख्या है।

(2*) कथन-1 सत्य है, कथन-2 सत्य है। कथन-2, कथन-1 की सही व्याख्या **नहीं** है।

(3) कथन-1 सत्य है, कथन-2 असत्य है।

(4) कथन-1 असत्य है, कथन-2 सत्य है।

Sol. **(2)**

$A' = A\ ,\ B' = A$

$P = A(BA)$

$P' = (A(BA))' = (BA)'\, A' = (A'B')\, A' = (AB)\, A = A(BA)$

$\therefore$ A(BA) is symmetric

similarly (AB) A is symmetric

Statement(2) is correct but not correct explanation of statement (1).

Hindi $A' = A\ ,\ B' = A$

$P = A(BA)$

$P' = (A(BA))' = (BA)'\, A' = (A'B')\, A' = (AB)\, A = A(BA)$

$\therefore$ A(BA) सममित है।

इसी प्रकार (AB) A भी सममित है।

कथन (2) सत्य है लेकिन कथन (1) की सही व्याख्या नहीं करता है।

5. The number of values of k for which the linear equations **[AIEEE 2011, I, (4, –1), 120]**

$4x + ky + 2z = 0$

$kx + 4y + z = 0$

$2x + 2y + z = 0$

posses a non-zero solution is :

k के उन मानों की संख्या जिनके लिए रैखिक समीकरण **[AIEEE 2011, I, (4, –1), 120]**

$4x + ky + 2z = 0$

$kx + 4y + z = 0$

$2x + 2y + z = 0$

के अशून्य मूल है, हैं :

(1) 3 (2*) 2 (3) 1 (4) zero शून्य

Sol. **(2)**

$$\Delta = \begin{vmatrix} 4 & k & 2 \\ k & 4 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0 \quad \Rightarrow \quad 8 - k(k-2) - 2(2k-8) = 0$$

$\Rightarrow\ 8 - k^2 + 2k - 4k + 16 = 0 \quad \Rightarrow \quad -k^2 - 2k + 24 = 0$

$\Rightarrow\ k^2 + 2k - 24 = 0 \quad \Rightarrow \quad (k+6)(k-4) = 0 \quad \Rightarrow \quad k = -6, 4$

Number of values of k is 2

Hindi
$$\Delta = \begin{vmatrix} 4 & k & 2 \\ k & 4 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0 \quad \Rightarrow \quad 8 - k(k-2) - 2(2k-8) = 0$$

$\Rightarrow\ 8 - k^2 + 2k - 4k + 16 = 0 \quad \Rightarrow \quad -k^2 - 2k + 24 = 0$

$\Rightarrow\ k^2 + 2k - 24 = 0 \quad \Rightarrow \quad (k+6)(k-4) = 0 \quad \Rightarrow \quad k = -6, 4$

अतः k के मानों की संख्या = 2

6. If $\omega \neq 1$ is the complex cube root of unity and matrix $H = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$, then H^{70} is equal to -

[AIEEE 2011, I, (4, –1), 120]

यदि $\omega \neq 1$ इकाई का एक सम्मिश्र घन मूल है तथा आव्यूह H = $\begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$ है, तो H^{70} बराबर है : **[AIEEE 2011, I, (4, –1), 120]**

(1) 0 (2) – H (3) H^2 (4*) H

Sol. $H^2 = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}\begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix}$

If $H^k = \begin{bmatrix} \omega^k & 0 \\ 0 & \omega^k \end{bmatrix}$, then $H^{k+1} = \begin{bmatrix} \omega^{k+1} & 0 \\ 0 & \omega^{k+1} \end{bmatrix}$

So by mathematical induction, गणितीय आगमन सिद्धान्त से

$H^{70} = \begin{bmatrix} \omega^{70} & 0 \\ 0 & \omega^{70} \end{bmatrix} = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} = H$

7. If the trivial solution is the only solution of the system of equations **[AIEEE 2011, II, (4, –1), 120]**

x – ky + z = 0
kx + 3y – kz = 0
3x + y – z = 0

then the set of all values of k is :

यदि एक समीकरण निकाय **[AIEEE 2011, II, (4, –1), 120]**

x – ky + z = 0
kx + 3y – kz = 0
3x + y – z = 0

का शून्य हल (trival solution) ही केवल हल है, तो k के सभी मानों का समुच्चय है :

(1*) R – {2, –3} (2) R – { 2 } (3) R – { –3 } (4) {2, –3}

Sol. **(1)**

x – ky + z = 0
kx + 3y – kz = 0
3x + y – z = 0

this system will have non trivial solution if इस निकाय का अशून्य हल (non-trival solution) होगा यदि

$$\begin{vmatrix} 1 & -k & 1 \\ k & 3 & -k \\ 3 & 1 & -1 \end{vmatrix} = 0$$

1(– 3 + k) + k(– k + 3k) + 1 (k– 9) = 0
$k - 3 + 2k^2 + k - 9 = 0$
$2k^2 + 2k - 12 = 0$
$k^2 + k - 6 = 0$
k = – 3, k = 2

so the system of equations will have only trivial solution when $k \in R - \{2, -3\}$

अतः समीकरण निकाय का शून्य हल विद्यमान होगा यदि $k \in R - \{2, -3\}$

8. **Statement - 1 :** Determinant of a skew-symmetric matrix of order 3 is zero. **[AIEEE 2011, II, (4, –1), 120]**

Statement - 2 : For any matrix A, det $(A)^T$ = det(A) and det (–A) = – det(A).

Where det (B) denotes the determinant of matrix B. Then :

(1) Both statements are true (2) Both statements are false
(3) Statement-1 is false and statement-2 is true. (4*) Statement-1 is true and statement-2 is false

कथन - 1 : कोटि 3 के विषम सममित आव्यूह के सारणिक का मान 0 है। **[AIEEE 2011, II, (4, –1), 120]**

कथन - 2 : किसी आव्यूह A के लिए det $(A)^T$ = det(A) तथा det (–A) = – det(A).

जहाँ det(B), आव्यूह B का सारणिक दर्शाता है :

(1) दोनों कथन सत्य है। (2) दोनों कथन असत्य है।
(3) कथन -1 असत्य है तथा कथन-2 सत्य है। (4*) कथन -1 सत्य है तथा कथन-2 असत्य है।

Sol. **(4)**

Statement-1 : Determinant of a skew sysmmetric matrix of odd order is zero

कथन-1 : विषम क्रम के विषम सममित आव्यूह के सारणिक का मान शून्य होता है।
Statement-2 : det (A^T) = det(A)
det(–A) = $(-1)^n$ det (A) where A is a n × n order matrix
कथन-2 : det (A^T) = det(A)
det(–A) = $(-1)^n$ det (A) जहाँ A , n × n क्रम का आव्यूह है।

9. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$. If u_1 and u_2 are column matrices such that $Au_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $Au_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then $u_1 + u_2$ is equal to : **[AIEEE-2012, (4, –1)/120]**

(1) $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ (2) $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ (3) $\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ (4*) $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

माना $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$ है। यदि u_1 तथा u_2 ऐसे स्तंभ आव्यूह हैं कि $Au_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ तथा $Au_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ है, तो $u_1 + u_2$ बराबर है: **[AIEEE-2012, (4, –1)/120]**

(1) $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ (2) $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ (3) $\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ (4*) $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

Sol. $A(u_1 + u_2) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$; $u_1 + u_2 = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}$ adj A ; $|A| = 1$

$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ Now $u_1 + u_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

10. Let P and Q be 3 × 3 matrices P ≠ Q. If $P^3 = Q^3$ and $P^2Q = Q^2P$, then determinant of $(P^2 + Q^2)$ is equal to : **[AIEEE-2012, (4, –1)/120]**

(1) – 2 (2) 1 (3*) 0 (4) – 1

माना P तथा Q, 3 × 3 आव्यूह हैं तथा P ≠ Q है। यदि $P^3 = Q^3$ तथा $P^2Q = Q^2P$ है, तो सारणिक $(P^2 + Q^2)$ बराबर है: **[AIEEE-2012, (4, –1)/120]**

(1) – 2 (2) 1 (3*) 0 (4) – 1

Sol. Subtracting $P^3 - P^2Q = Q^3 - Q^2P \Rightarrow P^2(P - Q) + Q^2(P - Q) = 0$
$(P^2 + Q^2)(P - Q) = 0$
If $|P^2 + Q^2| \neq 0$ then $P^2 + Q^2$ is invertible $\Rightarrow P - Q = O$ contradiction
Hence $|P^2 + Q^2| = 0$

Hindi. घटाने पर $P^3 - P^2Q = Q^3 - Q^2P \Rightarrow P^2(P - Q) + Q^2(P - Q) = 0 \Rightarrow (P^2 + Q^2)(P - Q) = 0$
यदि $|P^2 + Q^2| \neq 0$ तो $P^2 + Q^2$ प्रतिलोमिय है। $\Rightarrow P - Q = O$ विरोधाभास
अतः $|P^2 + Q^2| = 0$

11. The number of values of k, for which the system of equations : **[XI] JP & JP*[AIEEE - 2013, (4, – 1) 120]**

$(k + 1)x + 8y = 4k$
$kx + (k + 3)y = 3k - 1$

has no solution, is
(1) infinite (2*) 1 (3) 2 (4) 3
k के उन मानों की संख्या, जिनके लिए निम्न समीकरण निकाय
$(k + 1)x + 8y = 4k$
$kx + (k + 3)y = 3k – 1$
का कोई हल नहीं है, है– **[AIEEE - 2013, (4, – 1) 120]**
(1) अनन्त (2*) 1 (3) 2 (4) 3

Sol. **(2)**
$\frac{k+1}{k} = \frac{8}{k+3} \neq \frac{4k}{3k-1} \Rightarrow k^2 + 4k + 3 = 8k \Rightarrow k^2 - 4k + 3 = 0$
$k = 1, 3$
If $k = 1$ then $\frac{8}{1+3} \neq \frac{4.1}{2}$ False
And If $k = 3$ then $\frac{8}{6} \neq \frac{4.3}{9-1}$ True
therefore $k = 3$/ Hence only one value of k.

Hindi. **(2)**
$\frac{k+1}{k} = \frac{8}{k+3} \neq \frac{4k}{3k-1} \Rightarrow k^2 + 4k + 3 = 8k \Rightarrow k^2 - 4k + 3 = 0$
$k = 1, 3$
यदि $k = 1$ हो तब $\frac{8}{1+3} \neq \frac{4.1}{2}$ असत्य
और यदि $k = 3$ तब $\frac{8}{6} \neq \frac{4.3}{9-1}$ सत्य
इसलिये $k = 3$ अतः k का केवल एक मान ही संभव है।

12. If $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the adjoint of a 3 × 3 matrix A and |A| = 4, then α is equal to : **[XII] Only JP***
[AIEEE - 2013, (4, – 1) 120]

यदि $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ एक 3 × 3 आव्यूह A का सहखंडज है तथा |A| = 4 है तो α बराबर है :
[AIEEE - 2013, (4, – 1) 120]
(1) 4 (2*) 11 (3) 5 (4) 0

Sol. **(2)**
$|P| = 1(12 - 12) - \alpha(4 - 6) + 3(4 - 6) = 2\alpha - 6$
$|P| = |A|^2 = 16 \Rightarrow 2\alpha - 6 = 16 \Rightarrow \alpha = 11.$

13. If $\alpha, \beta \neq 0$ and $f(n) = \alpha^n + \beta^n$ and $\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix} = K(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$, then K is equal to
[JEE(Main) 2014, (4, – 1), 120]

यदि $\alpha, \beta \neq 0$ तथा $f(n) = \alpha^n + \beta^n$ तथा $\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix} = K(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$ है, तो K बराबर है–
[Matrices & Determinant]

[JEE(Main) 2014, (4, – 1), 120]

(1*) 1 (2) –1 (3) $\alpha\beta$ (4) $\frac{1}{\alpha\beta}$

Sol. Ans. (1)

$$\begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}^2$$

$= (1 - \alpha)^2 (1 - \beta)^2 (\alpha - \beta)^2$

14. If A is an 3 × 3 non-singular matrix such that AA′ = A′A and B = A^{-1}A′, then BB′ equals :

[JEE(Main) 2014, (4, – 1), 120]

यदि A एक ऐसा 3 × 3 व्युत्क्रमणीय आव्यूह है कि AA′ = A′A तथा B = A^{-1}A′ है, तो BB′ बराबर है–

[Matrices & Determinant]
[JEE(Main) 2014, (4, – 1), 120]

(1) B^{-1} (2) $(B^{-1})'$ (3) I + B (4*) I

Sol. Ans. (4)

$BB^T = B(A^{-1}A^T)^T$
$= B(A^T)^T (A^{-1})^T$
$= BA(A^{-1})^T$
$= A^{-1}A^TA(A^{-1})^T$
$= A^{-1}AA^T(A^{-1})^T$
$= IA^T(A^{-1})^T$
$= A^T(A^{-1})^T$
$= A^T(A^T)^{-1}$
$= I$

15. If A = $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is 3 × 3 identity matrix, then the ordered pair (a, b) is equal to :

(1) (2, – 1) (2) (–2, 1) (3) (2, 1) (4) (–2, – 1)

यदि A = $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ एक ऐसा आव्यूह है जो आव्यूह समीकरण $AA^T = 9I$, को संतुष्ट करता है जहाँ I, 3 × 3 का तत्समक आव्यूह है, तो क्रमित युग्म(a, b) का मान है : **[JEE(Main) 2015, (4, – 1), 120]**

(1) (2, – 1) (2) (–2, 1) (3) (2, 1) (4) (–2, – 1)

Ans. (4)

Sol. $AA^T = 9I$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ a + 4 + 2b = 0, 2a + 2 – 2b = 0, $a^2 + 4 + b^2 = 9$

$\Rightarrow$ a = – 2, b = – 1.

16. The set of all value of λ for which the system of linear equations :

$2x_1 - 2x_2 + x_3 = \lambda x_1$
$2x_1 - 3x_2 + 2x_3 = \lambda x_2$
$-x_1 + 2x_2 = \lambda x_3$

has a non-trivial solution, **[JEE(Main) 2015, (4, –1), 120]**

(1) is an empty set (2) is a singleton
(3) contains two elements (4) contains more than two elements

λ के सभी मानों का समुच्चय, जिनके लिए रेक्षिक समीकरण निकाय :

$2x_1 - 2x_2 + x_3 = \lambda x_1$

$2x_1 - 3x_2 + 2x_3 = \lambda x_2$

$- x_1 + 2x_2 = \lambda x_3$

का एक अतुच्छ हल है , **[JEE(Main) 2015, (4, –1), 120]**

(1) एक रिक्त समुच्चय है। (2) एक एकल समुच्चय है।

(3) में दो अवयव है। (4) में दो से अधिक अवयव है।

Ans. **(3)**

Sol. $\begin{vmatrix} \lambda-2 & 2 & -1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (\lambda - 2)(3\lambda + \lambda^2 - 4) - 2(-2\lambda + 2) - 1(4 - 3 - \lambda) = 0$

$\Rightarrow \quad (\lambda - 2)(\lambda^2 + 3\lambda - 4) + 4(\lambda - 1) + (\lambda - 1) = 0$

$\Rightarrow \quad (\lambda - 1)(\lambda^2 + 2\lambda - 8 + 5) = 0 \quad \Rightarrow \quad (\lambda - 1)(\lambda^2 + 2\lambda - 3) = 0$

Two elements $(\lambda - 1)^2 (\lambda + 3) = 0$

Hindi. $\begin{vmatrix} \lambda-2 & 2 & -1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (\lambda - 2)(3\lambda + \lambda^2 - 4) - 2(-2\lambda + 2) - 1(4 - 3 - \lambda) = 0$

$\Rightarrow \quad (\lambda - 2)(\lambda^2 + 3\lambda - 4) + 4(\lambda - 1) + (\lambda - 1) = 0$

$\Rightarrow \quad (\lambda - 1)(\lambda^2 + 2\lambda - 8 + 5) = 0 \quad \Rightarrow \quad (\lambda - 1)(\lambda^2 + 2\lambda - 3) = 0$

दो अवयव $(\lambda - 1)^2 (\lambda + 3) = 0$

17. The system of linear equations **[JEE(Main) 2016, (4, –1), 120]**

$x + \lambda y - z = 0$

$\lambda x - y - z = 0$

$x + y - \lambda z = 0$

has a non-trivial solution for :

(1) Exactly one value of λ. (2) Exactly two values of λ.

(3) Exactly three values of λ. (4) Infinitely many values of λ.

रैखिक समीकरण निकाय **[JEE(Main) 2016, (4, –1), 120]**

$x + \lambda y - z = 0$

$\lambda x - y - z = 0$

$x + y - \lambda z = 0$

का एक अतुच्छ हल होने के लिए:

(1) λ का तथ्यतः एक मान है। (2) λ के तथ्यतः दो मान हैं।

(3) λ के तथ्यतः तीन मान हैं। (4) λ के अनंत मान है।

Ans. **(3)**

Sol. $\begin{vmatrix} 1 & \lambda & -1 \\ \lambda & -1 & -1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$

$1(\lambda + 1) - \lambda(-\lambda^2 + 1) - 1(\lambda + 1) = 0$

$\lambda + 1 + \lambda^3 - \lambda - \lambda - 1 = 0$

$\lambda^3 - \lambda = 0$
Three values (3 माना)

18. If A = $\begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and A adj A = A A^T, then 5a + b is equal to **[JEE(Main) 2016, (4, –1), 120]**

यदि A = $\begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ तथा A adj A = A A^T हैं, तो 5a + b बराबर है : **[JEE(Main) 2016, (4, –1), 120]**

(1) 5 (2) 4 (3) 13 (4) – 1

Ans. **(1)**

Sol. $|A|I = AA^T$

$\Rightarrow (10a + 3b) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 5a & 3 \\ -b & 2 \end{bmatrix}$

$\Rightarrow 25a^2 + b^2 = 10a + 3b$ & $15a - 2b = 0$ & $10a + 3b = 13$

$\Rightarrow 10a + \dfrac{3.15a}{2} = 13$ $\Rightarrow 65a = 2 \times 13$

$\Rightarrow a = \dfrac{2}{5}$ $\Rightarrow 5a = 2$

$\Rightarrow 2b = 6$ $\Rightarrow b = 3$

$\therefore$ 5a + b = 5

19. It S is the set of distinct values of 'b' for which the following system of linear equations **[JEE(Main) 2017, (4, –1), 120]**

x + y + z = 1
x + ay + z = 1
ax + by + z = 0

has no solution, then S is :
(1) an empty set
(2) an infinite set
(3) a finite set containing two or more elements
(4) a singleton

यदि S 'b' की उन विभिन्न मानों का समुच्चय है जिनके लिए निम्न रैखिक समीकरण निकाय

x + y + z = 1
x + ay + z = 1
ax + by + z = 0

का कोई हल नहीं है, तो S :
(1) एक रिक्त समुच्चय है
(2) एक अपरिमित समुच्चय है
(3) एक परिमित समुच्च्य है जिसमें दो या अधिक अवयव है
(4) एक ही अवयव वाला समुच्चय है

Ans. **(4)**

Sol. $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & b & 1 \end{vmatrix} = -(a - 1)^2 = 0$

$\Rightarrow a = 1$

For a = 1 we have first two planes co-incident
a = 1 के लिए प्रथम दो समतल सम्पाती है
x + y + z = 1
ax + by + z = 0

For no solution these two are parallel

कोई हल नहीं के लिए दो समान्तर है।

$\frac{1}{a} = \frac{1}{b}\ \frac{1}{1} \Rightarrow a = 1\ ,\ b = 1$

20. If $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$, then adj $(3A^2 + 12A)$ is equal to

यदि $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$ है, तो adj $(3A^2 + 12A)$ बराबर है : **[JEE(Main) 2017, (4, –1), 120]**

(1) $\begin{bmatrix} 72 & -84 \\ -63 & 51 \end{bmatrix}$ (2) $\begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$ (3) $\begin{bmatrix} 51 & 84 \\ 63 & 72 \end{bmatrix}$ (4) $\begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$

Ans. **(2)**

Sol. $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$

$A^2 - 3A - 10I = 0$

$A^2 = 3A + 10I$

$3A^2 + 12A = 3(3A + 10I) + 12A = 21A + 30I$

$21A + 30I = \begin{bmatrix} 42 & -63 \\ -84 & 21 \end{bmatrix} + \begin{bmatrix} 30 & 0 \\ 0 & 30 \end{bmatrix} = \begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$

adj $(21A + 30I) = \begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$

21. If the system of linear equations **[JEE(Main) 2018, (4, –1), 120]**

$x + ky + 3z = 0$

$3x + ky - 2z = 0$

$2x + 4y - 3z = 0$

has a non-zero solution (x, y, z), then $\frac{xz}{y^2}$ is equal to :

यदि रैखिक समीकरण निकाय

$x + ky + 3z = 0$

$3x + ky - 2z = 0$

$2x + 4y - 3z = 0$

का एक शून्येतर हल (x, y, z) है, तो $\frac{xz}{y^2}$ बराबर है :

(1) – 30 (2) 30 (3) –10 (4*) 10

Sol. **(4)**

D = 0

$\begin{vmatrix} 1 & k & 3. \\ 3 & k & -2 \\ 2 & 4 & -3 \end{vmatrix} = 0$

$-3k + 8 - k(-9 + 4) + 3(12 - 2k) = 0$

$-3k + 8 + 5k + 36 - 6k = 0$

$-4k = -44 \qquad k = 11$

$x + 11y + 3z = 0$

$3x + 11y + 2z = 0$

$2x + 4y - 3z = 0$

$z = t$

$x + 11y = -3t$

$3x + 11y = 2t$

$2x = 5t$

$x = \frac{5t}{2}$

$$y = \frac{-3z - x}{11} = \frac{-3t - \frac{5t}{2}}{11} = \frac{-11t}{2 \times 11} = \frac{-t}{2}$$

$$\frac{xz}{y^2} = \frac{\frac{5t}{2} \times t}{\frac{t^2}{4}} = \frac{5}{2} \times 4 = 10$$

22. If $\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A + Bx)\,(x-A)^2$ then the ordered pair (A, B) is equal to :

यदि $\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A + Bx)\,(x-A)^2$ तो क्रमित युग्म (A, B) बराबर है : **[JEE(Main) 2018, (4, –1), 120]**

(1*) (–4, 5) (2) (4, 5) (3) (–4, –5) (4) (–4, 3)

Sol. **(1)**

$$\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A+Bx)\,(x-A)^2$$

$$\Rightarrow \begin{vmatrix} 5x-4 & 2x & 2x \\ 5x-4 & x-4 & 2x \\ 5x-4 & 2x & x-4 \end{vmatrix} = (A+Bx)\,(x-A)^2$$

$$\Rightarrow (5x-4)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x-4 & 2x \\ 1 & 2x & x-4 \end{vmatrix} = (A+Bx)\,(x-A)^2$$

$$\Rightarrow (5x-4)\begin{vmatrix} 1 & 2x & 2x \\ 0 & -x-4 & 0 \\ 1 & 0 & -x-4 \end{vmatrix} = (A+Bx)\,(x-A)^2$$

$(5x-4)\,(x+4)^2 = (A + Bx)\,(x-A)^2$

$A = -4, \; B = 5$

23. The system of linear equations **[JEE(Main) 2019, Online (09-01-19),P-1 (4, – 1), 120]**

$x + y + z = 2$

$2x + 3y + 2z = 5$

$2x + 3y + (a^2 - 1)z = a + 1$

(1) is inconsistent when a = 4

(2) has infinitely many solutions for a = 4

(3) is inconsistent when $|a| = \sqrt{3}$

(4) has a unique solution for $|a| = \sqrt{3}$

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$x + y + z = 2$

$2x + 3y + 2z = 5$

$2x + 3y + (a^2 - 1)z = a + 1$

(1) असंगत है जब a = 4

(2) के a = 4 के लिए अनन्त हल है।

(3) असंगत है जब $|a| = \sqrt{3}$

(4) का $|a| = \sqrt{3}$ के लिए मात्र एक हल है।

Ans. **(3)**

Sol. $D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 2 & 3 & a^2-1 \end{vmatrix}$ $\quad C_2 \to C_2 - C_1$ and तथा $C_3 \to C_3 - C_1$

$D = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & a^2-3 \end{vmatrix} = a^2 - 3$

Since if $a^2 \neq 3$ then only it will have unique solution. So at a = 4 unique solution exist.

चूंकि यदि $a^2 \neq 3$ तब यह अद्वितिय हल रखता है अतः a = 4 अद्वितिय हल रखता है।

24. If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then the matrix A^{-50} when $\theta = \frac{\pi}{12}$, is equal to :

यदि $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, तो आव्यूह A^{-50} जब $\theta = \frac{\pi}{12}$ बराबर है–

[JEE(Main) 2019, Online (09-01-19),P-1 (4, – 1), 120]

(1) $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ (2) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

(3) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ (4) $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

Ans. **(2)**

Sol. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, adj $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $\Rightarrow$ $A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$A^{-50} = \begin{bmatrix} \cos(-50\theta) & -\sin(-50\theta) \\ \sin(-50\theta) & \cos(-50\theta) \end{bmatrix}$

A^{-50} at $\theta = \frac{\pi}{12}$ is $\begin{bmatrix} \cos\frac{25\pi}{6} & \sin\frac{25\pi}{6} \\ -\sin\frac{25\pi}{6} & \cos\frac{25\pi}{6} \end{bmatrix}$

$= \begin{bmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} \\ -\sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

25. Let $d \in R$, and $A = \begin{bmatrix} -2 & 4+d & (\sin\theta - 2) \\ 1 & (\sin\theta)+2 & d \\ 5 & (2\sin\theta)-d & (-\sin\theta)+2+2d \end{bmatrix}$, $\theta \in [0, 2\pi]$. If the minimum value of det(A) is 8, then a value of d is :

माना $d \in R$ तथा $A = \begin{bmatrix} -2 & 4+d & (\sin\theta - 2) \\ 1 & (\sin\theta)+2 & d \\ 5 & (2\sin\theta)-d & (-\sin\theta)+2+2d \end{bmatrix}$, $\theta \in [0, 2\pi]$ । यदि det(A) का न्यूनतम मान 8 है, तो d का एक मान है : **[JEE(Main) 2019, Online (10-01-19),P-1 (4, – 1), 120]**

(1) –5 (2) $2(\sqrt{2}+2)$ (3) $2(\sqrt{2}+1)$ (4) –7

Ans. (1)

Sol. $|A| = \begin{vmatrix} -2 & 4+d & (\sin\theta-2) \\ 1 & (\sin\theta)+2 & d \\ 5 & (2\sin\theta)-d & (-\sin\theta)+2+2d \end{vmatrix}$

$= \begin{vmatrix} -2 & 4+d & (\sin\theta-2) \\ 1 & (\sin\theta)+2 & d \\ 1 & 0 & 0 \end{vmatrix}$ (New $R_3 = R_3 - 2R_2 + R_1$)

$= (4 + d)d - \sin^2\theta + 4 = (d + 2)^2 - \sin^2\theta$

Because minimum value of $|A| = 8 \Rightarrow (d + 2)^2 = 9 \Rightarrow d = 1$ or -5

चूंकि $|A|$ का न्यूनतम मान $= 8 \Rightarrow (d + 2)^2 = 9 \Rightarrow d = 1$ या -5

26. Let $A = \begin{pmatrix} 0 & 2q & r \\ p & q & -r \\ p & -q & r \end{pmatrix}$. If $AA^T = I_3$, then $|p|$ is

माना $A = \begin{pmatrix} 0 & 2q & r \\ p & q & -r \\ p & -q & r \end{pmatrix}$ यदि $AA^T = I_3$, तो $|p|$ बराबर है -

[JEE(Main) 2019, Online (11-01-19),P-1 (4, – 1), 120]

(1) $\frac{1}{\sqrt{5}}$ (2) $\frac{1}{\sqrt{3}}$ (3) $\frac{1}{\sqrt{6}}$ (4) $\frac{1}{\sqrt{2}}$

Ans. (4)

Sol. $AA^T = I \Rightarrow \begin{bmatrix} 0 & 2q & r \\ p & q & -r \\ p & -q & r \end{bmatrix} \begin{bmatrix} 0 & p & p \\ 2q & q & -q \\ r & -r & r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 4q^2+r^2 & 2q^2-r^2 & -2q^2+r^2 \\ 2q^2-r^2 & p^2+q^2+r^2 & p^2-q^2-r^2 \\ -2q^2+r^2 & p^2-q^2-r^2 & p^2+q^2+r^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= p^2 + q^2 + r^2 = 4q^2 + r^2 = 1$ and और $2q^2 - r^2 = 0,\ p^2 - q^2 - r^2 = 0$

Now अब $r^2 = 2q^2$ and और $r^2 + 4q^2 = 1 \Rightarrow q^2 = \frac{1}{6},\ r^2 = \frac{1}{3}$

Hence अतः $p^2 = \frac{1}{2} \Rightarrow |p| = \frac{1}{\sqrt{2}}$

27. Let $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$, $(\alpha \in R)$ such that $A^{32} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then a value of α is :

माना $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$, $(\alpha \in R)$ इस प्रकार है कि $A^{32} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, तो α का एक मान है–

(1) $\frac{\pi}{16}$ (2) $\frac{\pi}{32}$ (3) 0 (4) $\frac{\pi}{64}$

[JEE(Main) 2019, Online (08-04-19),P-1 (4, – 1), 120] [Matrices]

Ans. (4)

Sol. $A^2 = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$A^2 = \begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix}$

similarly we observe that इसी प्रकार $A^n = \begin{bmatrix} \cos n\alpha & -\sin n\alpha \\ \sin n\alpha & \cos n\alpha \end{bmatrix}$

hence अतः $\cos 32\alpha = 0$ and $\sin 32\alpha = 1$

$32\alpha = 2n\pi + \frac{\pi}{2}$

$\alpha = \frac{n\pi}{16} + \frac{\pi}{64}, n \in i$

28. If $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \ldots\ldots\ldots\ldots \begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 78 \\ 0 & 1 \end{bmatrix}$, then the inverse of $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ is :

यदि $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \ldots\ldots\ldots\ldots \begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 78 \\ 0 & 1 \end{bmatrix}$ है, तो $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ का व्युत्क्रम (inverse) है–

(1) $\begin{bmatrix} 1 & 0 \\ 12 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 0 \\ 13 & 1 \end{bmatrix}$ (3) $\begin{bmatrix} 1 & -12 \\ 0 & 1 \end{bmatrix}$ (4) $\begin{bmatrix} 1 & -13 \\ 0 & 1 \end{bmatrix}$

Ans. (4) **[JEE(Main) 2019, Online (09-04-19),P-1 (4, – 1), 120] [Matrix & Determinant]**

Sol. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \ldots\ldots \begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 78 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \frac{n(n-1)}{2} = 78 \Rightarrow n = 13$

$A = \begin{bmatrix} 1 & 13 \\ 0 & 1 \end{bmatrix}$ So $A^{-1} = \begin{bmatrix} 1 & -13 \\ 0 & 1 \end{bmatrix}$

29. If A is a symmetric matrix and B is skew symmetric matrix such that

$A + B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$, then AB is equal to

यदि एक सममित आव्यूह A तथा एक विषम सममित (symmetric matrix) आव्यूह B इस प्रकार है कि

$A + B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$, तो AB बराबर है –

(1) $\begin{bmatrix} -4 & 2 \\ 1 & 4 \end{bmatrix}$ (2) $\begin{bmatrix} 4 & -2 \\ -1 & -4 \end{bmatrix}$ (3) $\begin{bmatrix} 4 & -2 \\ 1 & -4 \end{bmatrix}$ (4) $\begin{bmatrix} -4 & -2 \\ -1 & 4 \end{bmatrix}$

Ans. (2) **[JEE(Main) 2019, Online (12-04-19),P-1 (4, – 1), 120] [Matrices & Determinant]**

Sol. $A + B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix} = P(\text{say})$

Now $A = \frac{P + P^T}{2}$ & $B = \frac{P - P^T}{2}$

So $A = \frac{1}{2}\left(\begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}\right) = \begin{bmatrix} 2 & 4 \\ 4 & -1 \end{bmatrix}$

$B = \frac{1}{2}\left(\begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

So $AB = \left(\begin{bmatrix} 2 & 4 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 4 & -2 \\ -1 & -4 \end{bmatrix}$

30. The total number of matrices $A = \begin{pmatrix} 0 & 2y & 1 \\ 2x & y & -1 \\ 2x & -y & 1 \end{pmatrix}$, (x, y∈R, x ≠ y) for which $A^TA = 3I_3$ is :

आव्यूहों $A = \begin{pmatrix} 0 & 2y & 1 \\ 2x & y & -1 \\ 2x & -y & 1 \end{pmatrix}$, (x, y∈R, x ≠ y) जिनके लिए $A^TA = 3I_3$ है, की कुल संख्या है–

(1) 3 (2) 4 (3) 6 (4) 2

Ans. **(2)** **[JEE(Main) 2019, Online (09-04-19),P-2 (4, – 1), 120]** **[Matrices & Determinant]**

Sol. If A′A = 3I, then AA′ = 3I

$$AA^T = \begin{bmatrix} 0 & 2y & 1 \\ 2x & y & -1 \\ 2x & -y & +1 \end{bmatrix}\begin{bmatrix} 0 & 2x & 2x \\ 2y & y & -y \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (4y^2+1) & (2y^2-1) & (-2y^2+1) \\ (2y^2-1) & (4x^2+y^{2+}1) & (4x^2-y^2-1) \\ (-2y^2+1) & (4x^2-y^2-1) & (4x^2+y^2+1) \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\therefore\ 4y^2 + 1 = 3 \quad \therefore\ y^2 = \frac{1}{2}$

$4x^2 + y^2 + 1 = 3$

$4x^2 + \frac{1}{2} + 1 = 3$

$\therefore\ 4x^2 = 3 - \frac{3}{2}$

$4x^2 = \frac{3}{2} \quad \therefore\ x^2 = \frac{3}{8}$

$(x, y) \equiv \left(\pm\sqrt{\frac{3}{8}}, \pm\frac{1}{\sqrt{2}}\right)$

(4 such matrices)

31. If the system of equations 2x + 3y – z = 0, x + ky – 2z = 0 and 2x – y + z = 0 has a non–trivial solution (x, y, z), then $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+k$ is equal to :

यदि समीकरण निकाय 2x + 3y – z = 0, x + ky – 2z = 0 तथा 2x – y + z = 0 का एक अतुच्छ (non–trivial) हल (x, y, z) है, तो $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+k$ बराबर है–

(1) –4 (2) $\frac{3}{4}$ (3) $\frac{1}{2}$ (4) $-\frac{1}{4}$

Ans. **(3)** **[JEE(Main) 2019, Online (09-04-19),P-2 (4, – 1), 120]** **[Matrices & Determinant]**

Sol. $\begin{vmatrix} 2 & 3 & -1 \\ 1 & k & -2 \\ 2 & -1 & 1 \end{vmatrix} = 0$

$R_1 \to R_1 - R_3$ and $R_3 \to R_3 - 2R_2$

$\Rightarrow \begin{vmatrix} 0 & 4 & -2 \\ 1 & k & -2 \\ 0 & -1-2k & 5 \end{vmatrix} = 0$

$\Rightarrow \quad k = \frac{9}{2}$

Equation 1 $\Rightarrow\ 2\frac{x}{y}+3-\frac{z}{y}=0$ ……(iv)

Equation 3 $\Rightarrow\ 2\frac{x}{y}-1+\frac{z}{y}=0$

adding these two equation we get $4\frac{x}{y}+2=0 \quad \Rightarrow \frac{x}{y}=-\frac{1}{2}$ …..(v)

Putting this value in equation (iv) we get $\frac{z}{y}=2$ ….(vi)

Equation (v) divided by equation (vi) $\Rightarrow \frac{x}{z}=-\frac{1}{4}$

So, $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+k = -\frac{1}{2} + \frac{1}{2} - 4 + \frac{9}{2} = \frac{1}{2}$

32. Let α be a root of equation $x^2 + x + 1 = 0$ and the matrix $A = \frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & \alpha & \alpha^2\\ 1 & \alpha^2 & \alpha^4\end{bmatrix}$, then the matrix A^{31} is equal to :

यदि समीकरण $x^2 + x + 1 = 0$ का एक मूल α है तथा आव्यूह $A = \frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & \alpha & \alpha^2\\ 1 & \alpha^2 & \alpha^4\end{bmatrix}$, है, तो आव्यूह A^{31} बराबर है :

(1) A^3 (2) A^2 (3) I_3 (4) A

Ans. **(1)** **[JEE(Main) 2020, Online (07-01-20),P-1 (4, – 1), 120]**

Sol. $A^2 = \frac{1}{3}\begin{bmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$

$\Rightarrow A^4 = I$

$\Rightarrow A^{30} = A^{28} \times A^3 = A^3$

33. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two 3×3 real matrices such that $b_{ij} = (3)^{(i+j-2)}a_{ji}$, where i, j = 1, 2, 3. If the determinant of B is 81, then the determinant of A is:

माना $A = [a_{ij}]$ तथा $B = [b_{ij}]$, 3×3 के दो वास्तविक आव्यूह इस प्रकार हैं कि $b_{ij} = (3)^{(i+j-2)}a_{ji}$, जहाँ i, j = 1, 2, 3. यदि B का सारणिक 81 है, तो A का सारणिक है: **[JEE(Main) 2020, Online (07-01-20),P-2 (4, –1), 120]**

(1) 1/9 (2) 1/81 (3) 3 (4) 1/3

Ans. **(1)**

Sol. $|B| = \begin{vmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{vmatrix} = \begin{vmatrix}3^0a_{11} & 3^1a_{12} & 3^2a_{13}\\ 3^1a_{21} & 3^2a_{22} & 3^3a_{23}\\ 3^2a_{31} & 3^3a_{32} & 3^4a_{33}\end{vmatrix} \Rightarrow 81 = 3^3.3.\ 3^2|A| \Rightarrow 3^4 = 3^6|A| \Rightarrow |A| = \frac{1}{9}$

High Level Problems (HLP)

1. If $a^2 + b^2 + c^2 = 1$, then prove that $\begin{vmatrix} a^2+(b^2+c^2)\cos\phi & ab(1-\cos\phi) & ac(1-\cos\phi) \\ ba(1-\cos\phi) & b^2+(c^2+a^2)\cos\phi & bc(1-\cos\phi) \\ ca(1-\cos\phi) & cb(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$

is independent of a, b, c

यदि $a^2 + b^2 + c^2 = 1$ हो, तो सिद्ध कीजिए कि सारणिक

$$\begin{vmatrix} a^2+(b^2+c^2)\cos\phi & ab(1-\cos\phi) & ac(1-\cos\phi) \\ ba(1-\cos\phi) & b^2+(c^2+a^2)\cos\phi & bc(1-\cos\phi) \\ ca(1-\cos\phi) & cb(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

का मान a, b, c से स्वतन्त्र है।

Sol. $a^2 + b^2 + c^2 = 1$

$$\Delta = \begin{vmatrix} a^2+(b^2+c^2)\cos\phi & ab(1-\cos\phi) & ac(1-\cos\phi) \\ ba(1-\cos\phi) & b^2+(c^2+a^2)\cos\phi & bc(1-\cos\phi) \\ ca(1-\cos\phi) & cb(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

$$\Delta = \frac{1}{abc}\begin{vmatrix} a^3+a(b^2+c^2)\cos\phi & b^2a(1-\cos\phi) & ac^2(1-\cos\phi) \\ ba^2(1-\cos\phi) & b^3+b(c^2+a^2)\cos\phi & bc^2(1-\cos\phi) \\ ca^2(1-\cos\phi) & cb^2(1-\cos\phi) & c^3+c(a^2+b^2)\cos\phi \end{vmatrix}$$

$$\frac{abc}{abc}\begin{vmatrix} a^2+(b^2+c^2)\cos\phi & b^2(1-\cos\phi) & c^2(1-\cos\phi) \\ a^2(1-\cos\phi) & b^2+(c^2+a^2)\cos\phi & c^2(1-\cos\phi) \\ a^2(1-\cos\phi) & b^2(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3$

$$= (a^2 + b^2 + c^2)\begin{vmatrix} 1 & b^2(1-\cos\phi) & c^2(1-\cos\phi) \\ 1 & b^2+(c^2+a^2)\cos\phi & c^2(1-\cos\phi) \\ 1 & b^2(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

संक्रिया $R_2 \to R_2 - R_1$ एवं $R_3 \to R_3 - R_1$ से

$$(a^2 + b^2 + c^2)\begin{vmatrix} 1 & b^2(1-\cos\phi) & c^2(1-\cos\phi) \\ 0 & (a^2+b^2+c^2)\cos\phi & 0 \\ 0 & 0 & (a^2+b^2+c^2)\cos\phi \end{vmatrix} = (a^2 + b^2 + c^2)^3 \cos^2\phi = \cos^2\phi$$

This is independent from a, b, c. यह a, b, c से स्वतन्त्र है।

2. If $a, b, c, x, y, z \in R$, then prove that,

यदि $a, b, c, x, y, z \in R$, तब सिद्ध कीजिए कि

$$\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} = . \begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix}$$

Sol. $$\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} = \begin{vmatrix} (a^2+x^2-2ax) & (b^2+x^2-2bx) & (c^2+x^2-2cx) \\ (a^2+y^2-2ay) & (b^2+y^2-2by) & (c^2+y^2-2cy) \\ (a^2+z^2-2az) & (b^2+z^2-2bz) & (c^2+z^2-2cz) \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ -2a & -2b & -2c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} \begin{vmatrix} a^2 & b^2 & c^2 \\ 2a & 2b & 2c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix}$$

3. If a_1, a_2, a_3 are distinct real roots of the equation $px^3 + px^2 + qx + r = 0$ such that $\begin{vmatrix} 1+a_1 & 1 & 1 \\ 1 & 1+a_2 & 1 \\ 1 & 1 & 1+a_3 \end{vmatrix} =$ 0, then Prove that $\frac{r}{p} < 0$

समीकरण $px^3 + px^2 + qx + r = 0$ के मूल यदि a_1, a_2, a_3 भिन्न-भिन्न वास्तविक हैं इसी प्रकार $\begin{vmatrix} 1+a_1 & 1 & 1 \\ 1 & 1+a_2 & 1 \\ 1 & 1 & 1+a_3 \end{vmatrix}$ $= 0$, हो तो सिद्ध कीजिए कि $\frac{r}{p} < 0$

Sol. $\mathbf{D} = \begin{vmatrix} 1+a_1 & 1 & 1 \\ 1 & 1+a_2 & 1 \\ 1 & 1 & 1+a_3 \end{vmatrix} = \begin{vmatrix} 1+a_1 & 0 & 1 \\ 1 & a_2 & 1 \\ 1 & -a_3 & 1+a_3 \end{vmatrix} = 0$

Applying $C_2 \rightarrow C_2 - C_3$

Expanding above determinant

$a_1a_2a_3 + a_1a_2 + a_2a_3 + a_3a_1 = 0$

Product of roots + $\Sigma a_1 a_2 = 0$

$a_1a_2 = q/p$, $a_1a_2a_3 = -r/p$

$-r/p + a/p = 0 \quad \Rightarrow \quad q = r$

$\Rightarrow \quad px^3 + px^2 + rx + r = 0 \quad \Rightarrow \quad (x + 1)(px^2 + r) = 0 \quad \Rightarrow \quad x^2 = -\frac{r}{p} \quad \Rightarrow \quad \frac{r}{p} < 0$

4. Prove that $\Delta = \begin{vmatrix} \beta\gamma & \beta\gamma'+\beta'\gamma & \beta'\gamma' \\ \gamma\alpha & \gamma\alpha'+\gamma'\alpha & \gamma'\alpha' \\ \alpha\beta & \alpha\beta'+\alpha'\beta & \alpha'\beta' \end{vmatrix} = (\alpha\beta' - \alpha'\beta)\,(\beta\gamma' - \beta'\gamma)\,(\gamma\alpha' - \gamma'\alpha)$

सिद्ध कीजिए कि : $\Delta = \begin{vmatrix} \beta\gamma & \beta\gamma'+\beta'\gamma & \beta'\gamma' \\ \gamma\alpha & \gamma\alpha'+\gamma'\alpha & \gamma'\alpha' \\ \alpha\beta & \alpha\beta'+\alpha'\beta & \alpha'\beta' \end{vmatrix} = (\alpha\beta' - \alpha'\beta)\,(\beta\gamma' - \beta'\gamma)\,(\gamma\alpha' - \gamma'\alpha)$

Sol. $\frac{1}{\alpha\beta\gamma} \begin{vmatrix} \alpha\beta\gamma & \alpha\beta\gamma'+\alpha\beta'\gamma & \alpha\beta'\gamma' \\ \alpha\beta\alpha & \beta\gamma\alpha'+\beta\gamma'\alpha & \beta\gamma'\alpha' \\ \alpha\beta\gamma & \alpha\beta'\gamma+\alpha'\beta\gamma & \alpha'\beta'\gamma \end{vmatrix}$

Multiplying R_1 by α, R_2 by β and R_3 by γ and dividing the whole determinant by $\alpha\beta\gamma$ we have

$$= \frac{1}{\alpha\beta\gamma} \cdot \alpha\beta\gamma \begin{vmatrix} 1 & \alpha\beta\gamma'+\alpha\beta'\gamma & \alpha\beta'\gamma' \\ 1 & \beta\gamma\alpha'+\beta\alpha\gamma' & \beta\gamma'\alpha' \\ 1 & \gamma\alpha\beta'+\gamma\alpha'\beta & \gamma\alpha'\beta' \end{vmatrix}$$

by $R_3 \rightarrow R_3 - R_1$

$R_2 \rightarrow R_2 - R_1$

$$= \begin{vmatrix} 1 & \alpha\beta\gamma'+\alpha\beta'\gamma & \alpha\beta'\gamma' \\ 0 & \gamma(\alpha'\beta-\alpha\beta') & \gamma'(\alpha'\beta-\alpha\beta') \\ 0 & \beta(\alpha'\gamma-\alpha\gamma') & \beta'(\alpha'\gamma-\alpha\gamma') \end{vmatrix} = (\alpha'\beta - \alpha\beta')\,(\alpha'\gamma - \alpha\gamma') \begin{vmatrix} 1 & \alpha\beta\gamma'+\alpha\beta'\gamma & \alpha\beta'\gamma' \\ 0 & \gamma & \gamma' \\ 0 & \beta & \beta' \end{vmatrix}$$

$= (\alpha'\beta - \alpha\beta')\,(\alpha'\gamma - \alpha\gamma')(\gamma\beta' - \beta\gamma')$

Sol. $\frac{1}{\alpha\beta\gamma}\begin{vmatrix} \alpha\beta\gamma & \alpha\beta\gamma'+\alpha\beta'\gamma & \alpha\beta'\gamma' \\ \alpha\beta\alpha & \beta\gamma\alpha'+\beta\gamma'\alpha & \beta\gamma'\alpha' \\ \alpha\beta\gamma & \alpha\beta'\gamma+\alpha'\beta\gamma & \alpha'\beta'\gamma \end{vmatrix}$

R_1 को α से , R_2 को β से तथा R_3 को γ से गुणा तथा सम्पूर्ण सारणिक को $\alpha\beta\gamma$ से विभाजित करने पर

$= \frac{1}{\alpha\beta\gamma}\cdot\alpha\beta\gamma\begin{vmatrix} 1 & \alpha\beta\gamma'+\alpha\beta'\gamma & \alpha\beta'\gamma' \\ 1 & \beta\gamma\alpha'+\beta\alpha\gamma' & \beta\gamma'\alpha' \\ 1 & \gamma\alpha\beta'+\gamma\alpha'\beta & \gamma\alpha'\beta' \end{vmatrix}$

संक्रिया $R_3 \to R_3 - R_1$
$R_2 \to R_2 - R_1$

$= \begin{vmatrix} 1 & \alpha\beta\gamma'+\alpha\beta'\gamma & \alpha\beta'\gamma' \\ 0 & \gamma(\alpha'\beta-\alpha\beta') & \gamma'(\alpha'\beta-\alpha\beta') \\ 0 & \beta(\alpha'\gamma-\alpha\gamma') & \beta'(\alpha'\gamma-\alpha\gamma') \end{vmatrix} = (\alpha'\beta-\alpha\beta')(\alpha'\gamma-\alpha\gamma')\begin{vmatrix} 1 & \alpha\beta\gamma'+\alpha\beta'\gamma & \alpha\beta'\gamma' \\ 0 & \gamma & \gamma' \\ 0 & \beta & \beta' \end{vmatrix}$

$= (\alpha'\beta-\alpha\beta')(\alpha'\gamma-\alpha\gamma')(\gamma\beta'-\beta\gamma')$

5. If $ax_1^2 + by_1^2 + cz_1^2 = d$ and $ax_2x_3 + by_2y_3 + cz_2z_3 = f$
$ax_2^2 + by_2^2 + cz_2^2 = d$ $\quad ax_3x_1 + by_3y_1 + cz_3z_1 = f$
$ax_3^2 + by_3^2 + cz_3^2 = d$ $\quad ax_1x_2 + by_1y_2 + cz_1z_2 = f$

then prove that $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}^2 = (d-f)^2\frac{d+2f}{abc}$, where a, b, c $\neq$ 0.

यदि $ax_1^2 + by_1^2 + cz_1^2 = d$ और $ax_2x_3 + by_2y_3 + cz_2z_3 = f$
$ax_2^2 + by_2^2 + cz_2^2 = d$ $\quad ax_3x_1 + by_3y_1 + cz_3z_1 = f$ हो, तो
$ax_3^2 + by_3^2 + cz_3^2 = d$ $\quad ax_1x_2 + by_1y_2 + cz_1z_2 = f$

सिद्ध कीजिए कि $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}^2 = (d-f)^2\frac{d+2f}{abc}$, जहाँ a, b, c $\neq$ 0.

Sol. $\begin{vmatrix} ax_1 & by_1 & cz_1 \\ ax_2 & by_2 & cz_2 \\ ax_3 & by_3 & cz_3 \end{vmatrix}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = abc\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}^2$; $\frac{1}{abc}\begin{vmatrix} d & f & f \\ f & d & f \\ f & f & d \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}^2$

L.H.S. $= \frac{1}{abc}(d+2f)\begin{vmatrix} 1 & f & f \\ 1 & d & f \\ 1 & f & d \end{vmatrix}$ $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

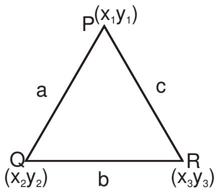
$\frac{1}{abc} = (d+2f)\begin{vmatrix} 1 & f & f \\ 0 & d-f & 0 \\ 0 & 0 & d-f \end{vmatrix} = \frac{(d-f)^2(d+2f)}{abc}$

6. If $\begin{matrix} (x_1-x_2)^2 + (y_1-y_2)^2 = a^2 \\ (x_2-x_3)^2 + (y_2-y_3)^2 = b^2 \\ (x_3-x_1)^2 + (y_3-y_1)^2 = c^2 \end{matrix}$, prove that $4\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = (a + b + c)(b + c - a)(c + a - b)(a + b - c)$.

यदि $\begin{matrix} (x_1-x_2)^2 + (y_1-y_2)^2 = a^2 \\ (x_2-x_3)^2 + (y_2-y_3)^2 = b^2 \\ (x_3-x_1)^2 + (y_3-y_1)^2 = c^2 \end{matrix}$ हो, तो सिद्ध कीजिए कि

$$4\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = (a + b + c)\,(b + c - a)\,(c + a - b)\,(a + b - c).$$

Sol.
$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = a^2$$
$$(x_2 - x_3)^2 + (y_2 - y_3)^2 = b^2$$
$$(x_3 - x_1)^2 + (y_3 - y_1)^2 = c^2$$

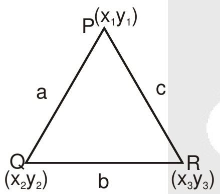


$$\Delta \text{ area of } \Delta\, PQR = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \Delta$$

$$2\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$4\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = 16\, s(s - a)\,(s - b)\,(s - c) = 2s\,(2s - 2a)\,(2s - 2b)\,(2s - 2c)$$
$$= (a + b + c)\,(b + c - a)\,(a + c - b)\,(a + b - c)$$

Hindi.
$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = a^2$$
$$(x_2 - x_3)^2 + (y_2 - y_3)^2 = b^2$$
$$(x_3 - x_1)^2 + (y_3 - y_1)^2 = c^2$$



Δ PQR का क्षेत्रफल $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \Delta$

$$2\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$4\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = 16\, s(s - a)\,(s - b)\,(s - c) = 2s\,(2s - 2a)\,(2s - 2b)\,(2s - 2c)$$
$$= (a + b + c)\,(b + c - a)\,(a + c - b)\,(a + b - c)$$

7. Let A = $\begin{bmatrix} \cos^{-1}x & \cos^{-1}y & \cos^{-1}z \\ \cos^{-1}y & \cos^{-1}z & \cos^{-1}x \\ \cos^{-1}z & \cos^{-1}x & \cos^{-1}y \end{bmatrix}$ such that |A| = 0, then find the maximum value of x + y + z

माना A = $\begin{bmatrix} \cos^{-1}x & \cos^{-1}y & \cos^{-1}z \\ \cos^{-1}y & \cos^{-1}z & \cos^{-1}x \\ \cos^{-1}z & \cos^{-1}x & \cos^{-1}y \end{bmatrix}$ इस प्रकार है कि |A| = 0, तो x + y + z का अधिकतम मान है।

Ans. 3

Sol. Let माना $a = \cos^{-1}x$, $b = \cos^{-1}y$, $c = \cos^{-1}z$,

$$|A| = 0 \Rightarrow \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a + b + c)\left[\frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}\right] = 0$$

$\Rightarrow$ a + b + c = 0 $\Rightarrow$ x = y = z = 1

8. If $y = \frac{u}{v}$, where u & v are functions of 'x', show that, $v^3 \frac{d^2y}{dx^2} = \begin{vmatrix} u & v & 0 \\ u' & v' & v \\ u'' & v'' & 2v' \end{vmatrix}$.

यदि $y = \frac{u}{v}$, जहाँ u और v , x' के फलन हो, तो प्रदर्शित कीजिए कि $v^3 \frac{d^2y}{dx^2} = \begin{vmatrix} u & v & 0 \\ u' & v' & v \\ u'' & v'' & 2v' \end{vmatrix}$.

Sol. $y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{vu'-uv'}{v^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{v^2(vu''+v'u'-u'v'-uv'')-2vv'(u'v-uv')}{v^4}$

$$v^3\frac{d^2y}{dx^2} = v(vu'' - uv'') - 2v'(u'v - uv') = \begin{vmatrix} v & 2v' & 0 \\ u' & u'' & u \\ v' & v'' & v \end{vmatrix} \Rightarrow v^3\frac{d^2y}{dx^2} = \begin{vmatrix} u & v & 0 \\ u' & v' & v \\ u'' & v'' & 2v' \end{vmatrix}$$

9. If α, β be the real roots of $ax^2 + bx + c = 0$ and $s_n = \alpha^n + \beta^n$, then prove that $as_n + bs_{n-1} + cs_{n-2} = 0$ for all $n \ge 2$, $n \in N$. Hence or otherwise prove that $\begin{vmatrix} 3 & 1+s_1 & 1+s_2 \\ 1+s_1 & 1+s_2 & 1+s_3 \\ 1+s_2 & 1+s_3 & 1+s_4 \end{vmatrix} \ge 0$ for all real a, b, c.

यदि α, β समीकरण $ax^2 + bx + c = 0$ के वास्तविक मूल है और $s_n = \alpha^n + \beta^n$ हो, तो सिद्ध कीजिए कि $as_n + bs_{n-1} + cs_{n-2} = 0$, जहाँ $n \ge 2$, $n \in N$ तथा इससे a, b, c के सभी वास्तविक मानों के लिए सिद्ध कीजिए कि $\begin{vmatrix} 3 & 1+s_1 & 1+s_2 \\ 1+s_1 & 1+s_2 & 1+s_3 \\ 1+s_2 & 1+s_3 & 1+s_4 \end{vmatrix} \ge 0$,

Sol. $ax^2 + bx + c = 0 \Rightarrow \alpha + \beta - \frac{b}{a} = \alpha\beta = \frac{c}{a}$

$s_n = \alpha^n + \beta^n \Rightarrow as_n + bs_{n-1} + cs_{n-2} = a(\alpha^n + \beta^n) + b(\alpha^{n-1} + \beta^{n-1}) + c(\alpha^{n-2} + \beta^{n-2})$

$$= a(\alpha^n + \beta^n) + b\left(\frac{\alpha^n}{\alpha}+\frac{\beta^n}{\beta}\right) + c\left(\frac{\alpha^n}{\alpha^2}+\frac{\beta^n}{\beta^2}\right) = \alpha^n\left(a+\frac{b}{\alpha}+\frac{c}{\alpha^2}\right) + \beta^n\left(a+\frac{b}{\beta}+\frac{c}{\beta^2}\right)$$

$= \alpha^{n-2}(\alpha^2 a + b\alpha + c) + \beta^{n-2}(a\beta^2 + b\beta + C) = 0 + 0 = 0$

and $\begin{vmatrix} 3 & 1+s_1 & 1+s_2 \\ 1+s_1 & 1+s_2 & 1+s_3 \\ 1+s_2 & 1+s_3 & 1+s_4 \end{vmatrix} = \begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}^2 \geq 0$

Hindi. $ax^2 + bx + c = 0 \quad \Rightarrow \quad \alpha + \beta - \frac{b}{a} = \alpha\beta = \frac{c}{a}$

$s_n = \alpha^n + \beta^n \quad \Rightarrow \quad as_n + bs_{n-1} + cs_{n-2} = a(\alpha^n + \beta^n) + b(\alpha^{n-1} + \beta^{n-1}) + c(\alpha^{n-2} + \beta^{n-2})$

$= a(\alpha^n + \beta^n) + b\left(\frac{\alpha^n}{\alpha} + \frac{\beta^n}{\beta}\right) + c\left(\frac{\alpha^n}{\alpha^2} + \frac{\beta^n}{\beta^2}\right) = \alpha^n\left(a + \frac{b}{\alpha} + \frac{c}{\alpha^2}\right) + \beta^n\left(a + \frac{b}{\beta} + \frac{c}{\beta^2}\right)$

$= \alpha^{n-2}(\alpha^2 a + b\alpha + c) + \beta^{n-2}(a\beta^2 + b\beta + C) = 0 + 0 = 0$

तथा $\begin{vmatrix} 3 & 1+s_1 & 1+s_2 \\ 1+s_1 & 1+s_2 & 1+s_3 \\ 1+s_2 & 1+s_3 & 1+s_4 \end{vmatrix} = \begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}^2 \geq 0$

10. Let $a > 0$, $d > 0$. Find the value of determinant $\begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{(a+d)} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{(a+2d)} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$

यदि $a > 0, d > 0$ हो, तो सारणिक $\begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{(a+d)} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{(a+2d)} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$ का मान ज्ञात कीजिए।

Ans. $\dfrac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$

Sol. $\Delta = \begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{(a+d)} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{(a+2d)} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$

by taking common $a(a + d)(a + 2d)$ from R_1 $(a + d)(a + 2d)(a + 3d)$ from R_2 and $(a + 2d)(a + 3d)(a + 4d)$ from R_3 in denominator, we have

$$\Delta = \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}\begin{vmatrix}(a+d)(a+2d) & (a+2d) & (a)\\(a+2d)(a+3d) & (a+3d) & (a+d)\\(a+3d)(a+4d) & (a+4d) & (a+2d)\end{vmatrix}$$

Let $\Delta' = \begin{vmatrix}(a+d)(a+2d) & (a+2d) & a\\(a+2d)2d & d & d\\(a+3d)(2d) & d & d\end{vmatrix}$

by $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$

$= \begin{vmatrix}(a+d)(a+2d) & (a+2d) & a\\(a+2d)2d & d & d\\2d^2 & 0 & 0\end{vmatrix}$ by $R_3 \to R_3 - R_2$

$\Delta' = 2d^3(a + 2d - a) = 4d^4$

$\therefore \quad \Delta' = \dfrac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$

Hindi. $\Delta = \begin{vmatrix}\frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)}\\\frac{1}{(a+d)} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)}\\\frac{1}{(a+2d)} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)}\end{vmatrix}$

R_1, R_2, R_3 में से क्रमशः $a(a + d)(a + 2d)$, $(a + d)(a + 2d)(a + 3d)$, $(a + 2d)(a + 3d)(a + 4d)$ उभयनिष्ठ लेने पर

$$\Delta = \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}\begin{vmatrix}(a+d)(a+2d) & (a+2d) & (a)\\(a+2d)(a+3d) & (a+3d) & (a+d)\\(a+3d)(a+4d) & (a+4d) & (a+2d)\end{vmatrix}$$

माना कि $\Delta' = \begin{vmatrix}(a+d)(a+2d) & (a+2d) & a\\(a+2d)2d & d & d\\(a+3d)(2d) & d & d\end{vmatrix}$

संक्रिया $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$

$= \begin{vmatrix}(a+d)(a+2d) & (a+2d) & a\\(a+2d)2d & d & d\\2d^2 & 0 & 0\end{vmatrix}$ by $R_3 \to R_3 - R_2$

$\Delta' = 2d^3(a + 2d - a) = 4d^4$

$\therefore \quad \Delta' = \dfrac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$

11. Let $\vec{a}_r = x_r\hat{i} + y_r\hat{j} + z_r\hat{k}$, $r = 1, 2, 3$ be three mutually perpendicular unit vectors, then find the value of $\begin{vmatrix}x_1 & x_2 & x_3\\y_1 & y_2 & y_3\\z_1 & z_2 & z_3\end{vmatrix}$

माना $\vec{a}_r = x_r\hat{i} + y_r\hat{j} + z_r\hat{k}$, , $r = 1, 2, 3$ तीन परस्पर लम्बवत् इकाई सदिश है तब $\begin{vmatrix}x_1 & x_2 & x_3\\y_1 & y_2 & y_3\\z_1 & z_2 & z_3\end{vmatrix}$ का मान है।

Ans. ± 1

Sol. $\Delta = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$; $\Delta^2 = \begin{vmatrix} x_1 & y_1 & z_2 \\ y_1 & y_2 & z_2 \\ z_1 & y_3 & z_3 \end{vmatrix} \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

$$= \begin{vmatrix} x_1^2 + y_1^2 + z_1^2 & x_1x_2 + y_1y_2 + z_1z_2 & x_1x_3 + y_1y_3 + z_1z_3 \\ x_1x_2 + y_2y_1 + z_2z_1 & x_2^2 + y_2^2 + z_2^2 & x_2x_3 + y_2y_3 + z_2z_3 \\ x_3x_1 + y_3y_1 + z_3z_1 & x_2x_3 + y_2y_3 + z_2z_3 & x_3^2 + y_3^2 + z_3^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \Rightarrow \Delta = \pm 1$$

12. If $\begin{vmatrix} x^k & x^{k+2} & x^{k+3} \\ y^k & y^{k+2} & y^{k+3} \\ z^k & z^{k+2} & z^{k+3} \end{vmatrix} = (x-y)(y-z)(z-x)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$, then find the value of k.

यदि $\begin{vmatrix} x^k & x^{k+2} & x^{k+3} \\ y^k & y^{k+2} & y^{k+3} \\ z^k & z^{k+2} & z^{k+3} \end{vmatrix} = (x-y)(y-z)(z-x)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$ हो तो k का मान ज्ञात कीजिए।

Ans. $k = -1$

Sol. $\begin{vmatrix} x^k & x^{k+2} & x^{k+3} \\ y^k & y^{k+2} & y^{k+3} \\ z^k & z^{k+2} & z^{k+3} \end{vmatrix} = x^ky^kz^k \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}$

$= (x-y)(y-z)(z-x)(xy + zy + zx)\, x^ky^kz^k$ at $k = -1$ पर

$= (x-y)(y-z)(z-x)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$

13. If the determinant $\begin{vmatrix} a+p & \ell + x & u+f \\ b+q & m+y & v+g \\ c+r & n+z & w+h \end{vmatrix}$ splits into exactly K determinants of order 3, each element of which contains only one term, then find the value of K =

यदि सारणिक $\begin{vmatrix} a+p & \ell + x & u+f \\ b+q & m+y & v+g \\ c+r & n+z & w+h \end{vmatrix}$ को 3 क्रम के K सारणिकों में प्रसारित कर दिया जाये जिसका प्रत्येक अवयव केवल एक पद रखता है तो K का मान ज्ञात करो।

Ans. 8

Sol. $\begin{vmatrix} a+p & \ell + x & u+f \\ b+q & m+y & v+g \\ c+r & n+z & w+h \end{vmatrix} = \begin{vmatrix} a+p & \ell + x & u \\ b+q & m+y & v \\ c+r & n+z & w \end{vmatrix} + \begin{vmatrix} a+p & \ell + x & f \\ b+q & m+y & g \\ c+r & n+z & h \end{vmatrix}$

$= \begin{vmatrix} a+p & \ell & u \\ b+q & m & v \\ c+r & n & w \end{vmatrix} + \begin{vmatrix} a+p & x & u \\ b+q & y & v \\ c+r & z & w \end{vmatrix} + \begin{vmatrix} a+p & \ell & f \\ b+q & m & g \\ c+r & n & h \end{vmatrix} + \begin{vmatrix} a+p & x & f \\ b+q & y & g \\ c+r & z & h \end{vmatrix}$

Now each det. may splits in 2 det. So, total determinants are 8.

$\begin{vmatrix} a+p & \ell + x & u+f \\ b+q & m+y & v+g \\ c+r & n+z & w+h \end{vmatrix} = \begin{vmatrix} a+p & \ell + x & u \\ b+q & m+y & v \\ c+r & n+z & w \end{vmatrix} + \begin{vmatrix} a+p & \ell + x & f \\ b+q & m+y & g \\ c+r & n+z & h \end{vmatrix}$

$= \begin{vmatrix} a+p & \ell & u \\ b+q & m & v \\ c+r & n & w \end{vmatrix} + \begin{vmatrix} a+p & x & u \\ b+q & y & v \\ c+r & z & w \end{vmatrix} + \begin{vmatrix} a+p & \ell & f \\ b+q & m & g \\ c+r & n & h \end{vmatrix} + \begin{vmatrix} a+p & x & f \\ b+q & y & g \\ c+r & z & h \end{vmatrix}$

अब प्रत्येक सारणिक 2 सारणिकों में विभक्त हो जाता है। अतः कुल 8 सारणिक बनते है।

14. If a, b, c are all different and $\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0$, then find the value of abc (ab + bc + ca) – (a + b + c).

यदि a, b, c सभी भिन्न-भिन्न है तथा $\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0$, तब abc (ab + bc + ca) – (a + b + c) का मान ज्ञात कीजिए।

Ans. 0

Sol. $\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0 \quad \Rightarrow \begin{vmatrix} a & a^3 & a^4 \\ b & b^3 & b^4 \\ c & c^3 & c^4 \end{vmatrix} = \begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix}$

abc (a – b) (b – c) (c – a) (ab + bc + ca) = (a – b) (b – c) (c – a) (a + b + c)

$\therefore$ abc (ab + bc + ca) = (a + b + c)

15. If a, b, c are complex numbers and $z = \begin{vmatrix} 0 & -b & -c \\ \bar{b} & 0 & -a \\ \bar{c} & \bar{a} & 0 \end{vmatrix}$ then show that z is purely imaginary.

यदि a, b, c सम्मिश्र संख्याएं है तथा $z = \begin{vmatrix} 0 & -b & -c \\ \bar{b} & 0 & -a \\ \bar{c} & \bar{a} & 0 \end{vmatrix}$ है तब दर्शाइये कि z एक विशुद्ध काल्पनिक है।

Sol. $z = \begin{vmatrix} 0 & -b & -c \\ \bar{b} & 0 & -a \\ \bar{c} & \bar{a} & 0 \end{vmatrix} = ba\bar{c} - ca\bar{b}$

Let $z_1 = ba\bar{c} = z_1 - \bar{z}_1 = 2\text{Im}(z_1)$ Purely Imaginary.

Hindi. $z = \begin{vmatrix} 0 & -b & -c \\ \bar{b} & 0 & -a \\ \bar{c} & \bar{a} & 0 \end{vmatrix} = ba\bar{c} - ca\bar{b}$

माना $z_1 = ba\bar{c} = z_1 - \bar{z}_1 = 2\text{Im}(z_1)$ विशुद्ध काल्पनिक

16. If $f(x) = \log_{10}x$ and $g(x) = e^{i\pi x}$ and $h(x) = \begin{vmatrix} f(x)g(x) & [f(x)]^{g(x)} & 1 \\ f(x^2)g(x^2) & [f(x^2)]^{g(x^2)} & 0 \\ f(x^3)g(x^3) & [f(x^3)]^{g(x^3)} & 1 \end{vmatrix}$, then find the value of h(10).

यदि $f(x) = \log_{10}x$ और $g(x) = e^{i\pi x}$ और $h(x) = \begin{vmatrix} f(x)g(x) & [f(x)]^{g(x)} & 1 \\ f(x^2)g(x^2) & [f(x^2)]^{g(x^2)} & 0 \\ f(x^3)g(x^3) & [f(x^3)]^{g(x^3)} & 1 \end{vmatrix}$, तब h(10) का मान है–

Ans. 0

Sol. $f(x) = \log_{10}x$, $g(x) = e^{i\pi x} = \cos\pi x + i\sin\pi x$

f(10) = 1 g(10) = 1

$h(10) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 3 & 3 & 1 \end{vmatrix} = 0$

17. If a, b, c, are real numbers, and D = $\begin{vmatrix} a & 1+2i & 3-5i \\ 1-2i & b & -7-3i \\ 3+5i & -7+3i & c \end{vmatrix}$ then show that D is purely real.

यदि a, b, c, वास्तविक संख्याएं है तथा D = $\begin{vmatrix} a & 1+2i & 3-5i \\ 1-2i & b & -7-3i \\ 3+5i & -7+3i & c \end{vmatrix}$ तब दर्शाइये कि D विशुद्ध वास्तविक है।

Sol. D = $\begin{vmatrix} a & 1+2i & 3-5i \\ 1-2i & b & -7-3i \\ 3+5i & -7+3i & c \end{vmatrix}$

By expanding above determinant we get
abc – 58a – 5c – 34b + 164 i.e., purely real.

Aliter : observe that D = D^T and $\bar{D} = D^T = D$ i.e. $\bar{D} = D$
i.e. D is real
$\bar{D}$ is a conjugate of D

Hindi D = $\begin{vmatrix} a & 1+2i & 3-5i \\ 1-2i & b & -7-3i \\ 3+5i & -7+3i & c \end{vmatrix}$

उपरोक्त सारणीक का प्रसार करने पर
abc – 58a – 5c – 34b + 164 जोकि वास्तविक है

वैकल्पिक : विश्लेषण से D = D^T तथा $\bar{D} = D^T = D$ अर्थात् $\bar{D} = D$ अर्थात् D वास्तविक है।
D का संयुग्मी $\bar{D}$ है।

18. If यदि $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ then तब find $\lim\limits_{n\to\infty} \frac{1}{n} A^n$ ज्ञात कीजिए।.

Ans. $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$

Sol. Let माना X = $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ $\Rightarrow$ $x^n = 0 \;\; \forall \; n \geq 2$ $\Rightarrow$ A = X + I

तब $A^n = (X + I)^n = {}^nC_0\, x^n + {}^nC_1 x^{n-1} I \,.................+ {}^nC_{n-2}\, x^2 + {}^nC_{n-1}\, x + {}^nC_n I = n\,x + I$

$= \begin{bmatrix} 1 & na \\ 0 & 1 \end{bmatrix}$

$\lim\limits_{n\to\infty} \frac{1}{n}\begin{bmatrix} 1 & na \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$

19. Let **P** = $\begin{bmatrix} \cos\frac{\pi}{9} & \sin\frac{\pi}{9} \\ -\sin\frac{\pi}{9} & \cos\frac{\pi}{9} \end{bmatrix}$ and α, β, γ be non-zero real numbers such that $\alpha p^6 + \beta p^3 + \gamma I$ is the zero matrix. Then find value of $(\alpha^2 + \beta^2 + \gamma^2)^{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$

माना **P** = $\begin{bmatrix} \cos\frac{\pi}{9} & \sin\frac{\pi}{9} \\ -\sin\frac{\pi}{9} & \cos\frac{\pi}{9} \end{bmatrix}$

तथा α, β, γ अशून्य वास्तविक संख्याऐं इस प्रकार है कि $\alpha p^6 + \beta p^3 + \gamma I$ शून्य आव्यूह हो, तो $(\alpha^2+\beta^2+\gamma^2)^{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$ का मान ज्ञात कीजिए।

Ans. 1

Sol. $$\mathbf{P}^2 = \begin{bmatrix} \cos\frac{\pi}{9} & \sin\frac{\pi}{9} \\ -\sin\frac{\pi}{9} & \cos\frac{\pi}{9} \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{9} & \sin\frac{\pi}{9} \\ -\sin\frac{\pi}{9} & \cos\frac{\pi}{9} \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\frac{\pi}{9} - \sin^2\frac{\pi}{9} & \cos\frac{\pi}{9}\sin\frac{\pi}{9} + \sin\frac{\pi}{9}\cos\frac{\pi}{9} \\ -\sin\frac{\pi}{9}\cos\frac{\pi}{9} - \cos\frac{\pi}{9}\sin\frac{\pi}{9} & -\sin^2\frac{\pi}{9} + \cos^2\frac{\pi}{9} \end{bmatrix} = \begin{bmatrix} \cos\frac{2\pi}{9} & \sin\frac{2\pi}{9} \\ -\sin\frac{2\pi}{9} & \cos\frac{2\pi}{9} \end{bmatrix}$$

$$\mathbf{P}^3 = \begin{bmatrix} \cos\frac{2\pi}{9} & \sin\frac{2\pi}{9} \\ -\sin\frac{2\pi}{9} & \cos\frac{2\pi}{9} \end{bmatrix}\begin{bmatrix} \cos\frac{\pi}{9} & \sin\frac{\pi}{9} \\ -\sin\frac{\pi}{9} & \cos\frac{\pi}{9} \end{bmatrix} ; \qquad \mathbf{P}^3 = \begin{bmatrix} \cos\frac{3\pi}{9} & \sin\frac{3\pi}{9} \\ -\sin\frac{3\pi}{9} & \cos\frac{3\pi}{9} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$\mathbf{P}^6 = \mathbf{P}^3 \cdot \mathbf{P}^3 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$\alpha\mathbf{P}^6 + \beta\mathbf{P}^3 + \alpha I = 0$

अर्थात $$\begin{bmatrix} -\frac{\alpha}{2} + \frac{\beta}{2} + \gamma & \frac{\sqrt{3}}{2}\alpha + \frac{\sqrt{3}}{2}\beta \\ -\frac{\sqrt{3}}{2}\alpha - \frac{\sqrt{3}}{2}\beta & -\frac{\alpha}{2} + \frac{\beta}{2} + \gamma \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\alpha = -\beta$ और $\beta = -\gamma \Rightarrow \alpha = \gamma \Rightarrow \alpha + \beta = 0$; $\beta + \gamma = 0$; $\alpha - \gamma = 0 \quad \Rightarrow \quad (\alpha^2 + \beta^2 + \gamma^2)^{o} = 1$

20. Consider an odd order square symmetric matrix $A = [a_{i\,j}]_{nxn}$. It's element in any row are 1, 2, ..., n in some order, then prove that a_{11}, a_{22}, ..., a_{nn} are numbers 1, 2, 3, ..., n in some order.

विषम क्रम के सममित वर्ग आव्यूह $A = [a_{i\,j}]_{nxn}$ के किसी भी पंक्ति मे अवयव 1, 2, ..., n किसी क्रम मे है तो सिद्ध कीजिए कि a_{11}, a_{22}, -, a_{nn} किसी क्रम मे 1, 2, 3, ..., n संख्याऐं है।

Sol. It is symmetric matrix, so each number should appear atleast once in diagonal (As each number in matrix will appear n times and n is odd). There are n diagonal places and n numbers so each number 1, 2, 3, ...,n will appear once in some order.

यह सममित आव्यूह है। अतः प्रत्येक संख्या कम से कम एक बार विकर्ण पर आनी चाहिये। (जैसा कि प्रत्येक संख्या आव्यूह मे n बार आती है तथा n विषम है) यहाँ विकर्ण पर n स्थान होगें तथा n संख्याएँ है जिससे प्रत्येक संख्या 1, 2, 3, ...,n विकर्ण पर किसी भी क्रम में एक बार तो आएगी।

21. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ and $\mathbf{C} = 3\mathbf{A} + 7\mathbf{B}$

Prove that

(i) $(\mathbf{A} + \mathbf{B})^{2013} = \mathbf{A}^{2013} + \mathbf{B}^{2013}$

(ii) Prove that $\mathbf{A}^n = 3^{n-1}\,\mathbf{A}$; $\mathbf{B}^n = 3^{n-1}\,\mathbf{B}$; $\mathbf{C}^n = 3^{2n-1}\,\mathbf{A} + 7.21^{n-1}\,\mathbf{B}$.

Hindi. माना $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

और $\mathbf{C} = 3\mathbf{A} + 7\mathbf{B}$

सिद्ध कीजिए कि

(i) $(\mathbf{A} + \mathbf{B})^{2013} = \mathbf{A}^{2013} + \mathbf{B}^{2013}$

(ii) सिद्ध कीजिए कि $A^n = 3^{n-1}A$; $B^n = 3^{n-1}B$; $C^n = 3^{2n-1}A + 7.21^{n-1}B$.

Sol. **(i)** Note that $\because$ $AB = BA = 0$

$\therefore$ $(A + B)^{2013} = A^{2013} + B^{2013}$ (By using Binomial Theorem)

(ii) $A^2 = 3A$ and so is $B^2 = 3B$

Using induction it can be proved that $A^n = 3^{n-1}A$ and $B^n = 3^{n-1}B$.

$C = 3A + 7B$

$C^n = (3A)^n + (7B)^n = 3^{2n-1}A + 7.21^{n-1}B$

Hindi. **(i)** $\because$ $AB = BA = 0$

$\therefore$ $(A + B)^{2013} = A^{2013} + B^{2013}$ (द्विपद प्रमेय के उपयोग से)

(ii) $A^2 = 3A$ इसलिए $B^2 = 3B$

आगमन सिद्धान्त की सहायता से सिद्ध कर सकते है $A^n = 3^{n-1}A$ तथा $B^n = 3^{n-1}B$.

$C = 3A + 7B$

$C^n = (3A)^n + (7B)^n = 3^{2n-1}A + 7.21^{n-1}B$

22. Let **'A'** is (4×4) matrix such that sum of elements in each row is 1. Find out sum of all the elements in A^{10}.

माना **'A'**. (4×4) आव्यूह इस प्रकार है कि इसकी प्रत्येक पंक्ति के अवयवों का योगफल 1 है, तो A^{10} के सभी अवयवों का योगफल ज्ञात कीजिए।

Ans. 4

Sol. Given $\sum_{k=1}^{4} a_{ik} = 1 \quad \forall\ i \in \{1,2,3,4\}$

Let's consider $B = A^2$ $\quad B = [b_{ij}]_{4\times4}$

$$b_{ij} = \sum_{k=1}^{4} a_{ik} \cdot a_{kj}$$

Sum of elements of i^{th} row in A^2 is

$$\sum_{J=1}^{4} b_{ij} = \sum_{j=1}^{4}\sum_{k=1}^{4} a_{ik}.a_{kj} = \sum_{K=1}^{4}\sum_{J=1}^{4} a_{ik}.a_{kj} = \sum_{K=1}^{4} a_{ik}\sum_{j=1}^{4} a_{kj}$$

$\because \sum_{j=1}^{4} a_{kj} = 1 = \sum_{K=1}^{4} a_{ik} = 1 \Rightarrow$ Sum of elements in a row of A^2 is 1.

$\Rightarrow$ Similarly sum of elements in a row of A^{10} is $1 \Rightarrow$ Sum of all elements = 4

Hindi दिया है $\sum_{k=1}^{4} a_{ik} = 1 \quad \forall\ i \in \{1,2,3,4\}$

माना $B = A^2$ $\quad B = [b_{ij}]_{4\times4}$ $\quad \Rightarrow \quad \sum_{k=1}^{4} b_{ij} = a_{ik} \cdot a_{kj}$

A^2 मे i^{th} पंक्ति के अवयवों का योगफल 1 है

$$\sum_{J=1}^{4} b_{ij} = \sum_{j=1}^{4}\sum_{k=1}^{4} a_{ik}.a_{kj} = \sum_{K=1}^{4}\sum_{J=1}^{4} a_{ik}.a_{kj} = \sum_{K=1}^{4} a_{ik}\sum_{j=1}^{4} a_{kj}$$

$\because \sum_{j=1}^{4} a_{kj} = 1 = \sum_{K=1}^{4} a_{ik} = 1 \Rightarrow A^2$ की पंक्ति के अवयवों का योगफल है

$\Rightarrow$ इसी प्रकार A^{10} की पंक्ति के अवयवों का योगफल 1 है

$\Rightarrow$ सभी अवयवों का योगफल 4 है

23. Let $A = \begin{bmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{bmatrix}$, then prove that A^{-1} exists if $3x + \lambda \neq 0$, $\lambda \neq 0$

माना $A = \begin{bmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{bmatrix}$, तब दर्शाइये कि A^{-1} विद्यमान है यदि $3x + \lambda \neq 0, \lambda \neq 0$

Sol. $A = \begin{bmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{bmatrix}$; $|A| = (3x+\lambda)\begin{vmatrix} 1 & x & x \\ 1 & x+\lambda & x \\ 1 & x & x+\lambda \end{vmatrix}$

by $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$= (3x+\lambda)\begin{vmatrix} 1 & x & x \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} = (3x+\lambda)(\lambda^2)$

$|A| \neq 0$ for non-singular matrix $\therefore$ $3x + \lambda \neq 0, \lambda \neq 0$

Hindi. $A = \begin{bmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{bmatrix}$; $|A| = (3x+\lambda)\begin{vmatrix} 1 & x & x \\ 1 & x+\lambda & x \\ 1 & x & x+\lambda \end{vmatrix}$

संक्रिया $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ से

$= (3x+\lambda)\begin{vmatrix} 1 & x & x \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} = (3x+\lambda)(\lambda^2)$

व्युत्क्रमणीय मैट्रिक्स के लिए $|A| \neq 0$

$\therefore$ $3x + \lambda \neq 0, \lambda \neq 0$

24. Prove that if A and B are $n \times n$ matrices, then $\det(I_n - AB) = \det(I_n - BA)$.

सिद्ध करो कि यदि A तथा B, $n \times n$ क्रम की आव्यूह है, तब $\det(I_n - AB) = \det(I_n - BA)$.

Sol. Let us assume first that B is invertible. Then $I_n - AB = B^{-1}(I_n - AB)B$ and hence $\det(I_n - AB) = \det(B^{-1})\det(I_n - BA)\det B = \det(I_n - BA)$.

If B is not invertible, consider instead the matrix $B_x = xI_n + B$. Since $\det(xI_n + B)$ is a polynomial in x, the matrices B_x are invertible for all but finitely many values of x. So we can use the first part of the proof and conclude that $\det(I_n - AB_x) = \det(I_n - B_xA)$ for all except finitely many values of x. But these two determinants are polynomials in x, which are equal for infinitely many values of x, so they must always be equal. In particular, for $x = 0$, $\det(I_n - AB) = \det(I_n - BA)$.

Hindi. माना कि B व्युत्क्रमणीय आव्यूह तब $I_n - AB = B^{-1}(I_n - AB)B$ तथा

$\det(I_n - AB) = \det(B^{-1})\det(I_n - BA)\det B = \det(I_n - BA)$.

यदि B प्रतिलोमीय नहीं है माना कि आव्यूह $B_x = xI_n + B$ चूँकि $(xI_n + B)$, x में बहुपद है, आव्यूह B_x सभी लेकिन x के सीमित अनेक मानों के लिए प्रतिलोमीय है। अतः हम सत्यापन का प्रथम भाग का प्रयोग कर सकते है तथा निष्कर्ष निकालते है कि सभी लेकिन x के सीमित संख्या में मान को छोडकर सभी मानों के लिए $\det(I_n - AB_x) = \det(I_n - B_xA)$ परन्तु ये दोनों सारणिक x में बहुपद जो कि x के अनन्त मानों के लिए बराबर है अतः यह हमेशा बराबर होना चाहिए। विशेषतः $x = 0$ के लिए $\det(I_n - AB) = \det(I_n - BA)$.

25. Let A be an $n \times n$ matrix such that $A^n = \alpha A$ where α is a real number different from 1 and -1. Prove that the matrix $A + I_n$ is invertible.

माना A एक $n \times n$ क्रम का आव्यूह इस प्रकार है कि $A^n = \alpha A$ जहाँ α , -1 और 1 से भिन्न एक वास्तविक संख्या है। सिद्ध करो कि आव्यूह $A + I_n$ प्रतिलोमीय है।

Sol. Let $B = A + I_n$. Since $A = B - I_n$, the condition $A^n = \alpha A$ can be written in the form $(B - I_n)^n = \alpha(B - I_n)$. This implies

$$B^n - \binom{n}{1}B^{n-1} + \binom{n}{2}B^{n-2} + \ldots\ldots + (-1)^n I_n = \alpha B - \alpha I_n$$

or $\quad B^n - \binom{n}{1} B^{n-1} + \binom{n}{2} B^{n-2} + ... + (-1)^{n-1} B - \alpha B = -\alpha I_n - (-1)^n I_n.$

Thus $\quad B = \left(B^{n-1} - \binom{n}{1} B^{n-2} + \binom{n}{2} B^{n-3} + ... + (-1)^{n-1} I_n - \alpha I_n\right) ((-1)^{n+1} - \alpha)\, I_n,$

which clearly means that B is invertible, since $(-1)^{n+1} - \alpha \neq 0$.

Hindi माना $B = A + I_n$ अतः $A = B - I_n$ प्रतिबन्ध $A^n = \alpha A$ को $(B - I_n)^n = \alpha (B - I_n)$ के रूप में लिखा जा सकता है।

$$B^n - \binom{n}{1} B^{n-1} + \binom{n}{2} B^{n-2} + ... + (-1)^n I_n = \alpha B - \alpha I_n$$

या

$$B^n - \binom{n}{1} B^{n-1} + \binom{n}{2} B^{n-2} + ... + (-1)^{n-1} B - \alpha B = -\alpha I_n - (-1)^n I_n.$$

अब $\quad B = \left(B^{n-1} - \binom{n}{1} B^{n-2} + \binom{n}{2} B^{n-3} + ... + (-1)^{n-1} I_n - \alpha I_n\right) ((-1)^{n+1} - \alpha)\, I_n,$

स्पष्टतया B प्रतिलोमीय है चूँकि $(-1)^{n+1} - \alpha \neq 0$.

26. Let p and q be real numbers such that $x^2 + px + q \neq 0$ for every real number x. Prove that if n is an odd positive integer, then $X^2 + pX + qI_n \neq 0_n$ for all real matrices X of order $n \times n$.

माना p तथा q वास्तविक संख्याएँ इस प्रकार है कि प्रत्येक वास्तविक संख्या x के लिए $x^2 + px + q \neq 0$ सिद्ध करो कि यदि n एक विषम धनात्मक पूर्णांक है तो $n \times n$ क्रम की सभी वास्तविक आव्यूहों के लिए $X^2 + pX + qI_n \neq 0_n$

Sol. Assume $X^2 + pX + qI_n = O_n$ for some $n \times n$ matrix X. This equality can be written in the form

$$\left(X + \frac{p}{2} I_n\right)^2 = \frac{p^2 - 4q}{4} I_n.$$

Taking determinants on both sides and using the fact that det(AB) = det A . det B we obtain

$$\left(\det\left(X + \frac{p}{2} I_n\right)\right)^2 = \left(\frac{p^2 - 4q}{4}\right)^n$$

This left side is nonnegative, and the right side is strictly negative by hypothesis. This contradiction ends the proof.

Hindi. माना कि कुछ $n \times n$ क्रम के आव्यूह x के लिए $X^2 + pX + qI_n = O_n$ यह समिका निम्न रूप $= I_n$ में लिखी जा सकती है।

$$\left(X + \frac{p}{2} I_n\right)^2 = \frac{p^2 - 4q}{4} I_n.$$

दोनों ओर सारणिक लेने पर तथा det(AB) = det A . det B का प्रयोग करने पर

हमें प्राप्त होता है $\left(\det\left(X + \frac{p}{2} I_n\right)\right)^2 = \left(\frac{p^2 - 4q}{4}\right)^n$ यहाँ बाँयां पक्ष अऋणात्मक है तथा दाँया पक्ष परिकल्पना से नियमबद्ध ऋणात्मक है जो कि परस्पर विरोधाभास है।

27. Let **A, B, C** be three 3×3 matrices with real entries. If **BA** + **BC** + **AC** = I and det(**A + B**) = 0 then find the value of det(**A + B + C – BAC**).

माना **A, B, C** तीन 3×3 क्रम के वास्तविक अवयवों का आव्यूह है। यदि **BA** + **BC** + **AC** = I तथा det(**A + B**) = 0 तब det(**A + B + C – BAC**) का मान ज्ञात कीजिए।

Ans. 0

Sol. $BA = I - AC - BC \quad \Rightarrow \quad BAC = C - AC^2 - BC^2$

$\Rightarrow \quad C - BAC = AC^2 + BC^2 \quad \Rightarrow \quad C - BAC = (A + B)C^2$

$\Rightarrow$ **A + B + C – BAC = A + B + (A + B) C² = (A + B) (I + C²)**

$\therefore$ det(**A + B + C – BAC**) = det((**A + B**) (**I + C²**)) = det(**A + B**) · det(**I + C²**) = 0

28. If $|z_1| = |z_2| = 1$, then prove that $\begin{bmatrix} z_1 & -z_2 \\ \bar{z}_2 & \bar{z}_1 \end{bmatrix}^{-1} \begin{bmatrix} \bar{z}_1 & z_2 \\ -\bar{z}_2 & z_1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

यदि $|z_1| = |z_2| = 1$, हो तो सिद्ध कीजिए कि $\begin{bmatrix} z_1 & -z_2 \\ \bar{z}_2 & \bar{z}_1 \end{bmatrix}^{-1} \begin{bmatrix} \bar{z}_1 & z_2 \\ -\bar{z}_2 & z_1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

Sol. $\begin{bmatrix} z_1 & -z_2 \\ \bar{z}_2 & \bar{z}_1 \end{bmatrix}^{-1} \begin{bmatrix} \bar{z}_1 & z_2 \\ -\bar{z}_2 & z_1 \end{bmatrix}^{-1} = \left(\begin{bmatrix} \bar{z}_1 & z_2 \\ -\bar{z}_2 & z_1 \end{bmatrix} \begin{bmatrix} z_1 & -z_2 \\ \bar{z}_2 & \bar{z}_1 \end{bmatrix}\right)^{-1}$ $(\because A^{-1} . B^{-1} = (BA)^{-1})$

$= \begin{bmatrix} z_1\bar{z}_1 + z_2\bar{z}_2 & 0 \\ 0 & z_2\bar{z}_2 + z_1\bar{z}_1 \end{bmatrix}^{-1} = \begin{bmatrix} |z_1|^2 + |z_2|^2 & 0 \\ 0 & |z_1|^2 + |z_2|^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

29. If A and B are two square matrices such that $B = -A^{-1} BA$, then show that $(A + B)^2 = A^2 + B^2$

यदि A और B दो वर्ग आव्यूह इस प्रकार है कि $B = -A^{-1} BA$, तब दर्शाइये कि $(A + B)^2 = A^2 + B^2$

Sol. $B = -A^{-1} BA \Rightarrow AB = -AA^{-1} BA$

$AB = -BA \Rightarrow AB + BA = 0 \quad (A + B)^2 = A^2 + B^2$

30. If the system of equations $x = cy + bz$, $y = az + cx$ and $z = bx + ay$ has a non-zero solution and at least one of a, b, c is a proper fraction, prove that $a^3 + b^3 + c^3 < 3$ and $abc > -1$.

यदि समीकरण निकाय $x = cy + bz$, $y = az + cx$ तथा $z = bx + ay$ के अशून्य हल हो तथा a, b, c मे कम से कम एक उचित भिन्न हो, तो सिद्ध कीजिए कि $a^3 + b^3 + c^3 < 3$ तथा $abc > -1$.

Sol. We have given that system has non-trival solution दिये गये समीकरण निकाय के अशून्य हल

$\Rightarrow \begin{vmatrix} 1 & -c & -b \\ -c & 1 & -a \\ -b & -a & 1 \end{vmatrix} = 0 \Rightarrow 1(1 - a^2) + c(-c - ab) - b(ac + b) = 0$

$\Rightarrow 1 - 2abc - a^2 - b^2 - c^2 = 0 \Rightarrow (a + bc)^2 = (1 - b^2)(1 - c^2)$

Similarly इस प्रकार $(b + ac)^2 = (1 - a^2)(1 - b^2)$, $(c + ab)^2 = (1 - a^2)(1 - b^2)$

Hence अत: $(1 - a^2), (1 - b^2), (1 - c^2)$

All have same sign. Since atleast one of them is

सभी समान चिन्ह के है चूंकि इसमें से कम से कम $1 - a^2 > 0$, $1 - b^2 > 0$, $1 - c^2 > 0$ होगा

$\Rightarrow a^2 + b^2 + c^2 < 3 \Rightarrow 1 - 2abc < 3 \Rightarrow abc > -1$

31. If $D = \text{diag}\{d_1, d_2, ..., d_n\}$, then prove that $f(D) = \text{diag}\{f(d_1), f(d_2), ..., f(d_n)\}$, where f(x) is a polynomial with scalar coefficient.

यदि $D = \text{diag}\{d_1, d_2, ..., d_n\}$, तब सिद्ध कीजिए कि $f(D) = \text{diag}\{f(d_1), f(d_2), ..., f(d_n)\}$, जहाँ f(x) अदिश गुणांको के साथ बहुपदीय है।

Sol. Let माना $f(x) = a_0 + a_1x + a_2x^2 + ... + a_n x^n$

$\Rightarrow f(D) = a_0I + a_1D + a_2D^2 + ... + a_nD^n$

$= a_0 \times \text{diag}\{1, 1, ..., 1\}$

$+ a_1 \times \text{diag}\{d_1, d_2, ..., d_n\}$

$+ a_2 \times \text{diag}\{d_1^2, d_2^2, ..., d_n^2\}$

....

$+ a_n \times \text{diag}\{d_1^n, d_2^n, ..., d_n^n\}$

$= \text{diag}\{a_0 + a_1d_1 + a_2d_1^2 .... + a_nd_1^n, ..., a_0 + a_1d_n + a_nd_n^2 + ... + a_nd_n^n\}$

$= \text{diag}\{f(d_1), f(d_2), ... f(d_n)\}$.

32. Given the matrix **A** = $\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$ and **X** be the solution set of the equation A^x = **A**, where x ∈ **N** – {1}. Evaluate $\Pi \frac{x^3+1}{x^3-1}$; where the continued product extends ∀ x ∈ **X.**

एक आव्यूह **A** = $\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$ तथा समीकरण A^x = **A** जहाँ x ∈ **N** – {1} का हल समूच्चय **X** हो, तो $\Pi \frac{x^3+1}{x^3-1}$; जहाँ ∀ x ∈ **X** के लिए सतत् गुणफल है, परिकलन कीजिए।

Sol. $A^2 = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} = A$

⇒ $A^x = A$ ∀ x = {2, 3, 4, ...}

$$\prod_{x=2}^{\infty} \frac{x^3+1}{x^3-1} = \prod_{x=2}^{\infty} \frac{(x+1)}{(x-1)} \frac{(x^2-x+1)}{(x^2+x+1)}$$

$$= \lim_{n\to\infty} \left(\frac{3}{1}.\frac{4}{2}.\frac{5}{3}.\frac{6}{4}............\frac{n}{(n-2)}\frac{(n+1)}{n-1}\right)\left(\frac{3}{7}.\frac{7}{13}.\frac{13}{21}............\frac{n^2-n+1}{n^2+n+1}\right)$$

$$= \lim_{n\to\infty} \frac{n(n+1)}{2}.\frac{3}{(n^2+n+1)} = \frac{3}{2}$$ **Ans.**

Comprehension (Q. NO. 33 to 35)

Any non-zero vector, X, is said to be characteristic vector of a matrix A, if there exist a number λ such that AX = λX. And then λ is said to be a charactristic root of the matrix A corresponding to the characteristic vector X and vice versa.

Also AX = λX ⇒ (A – λI)X = 0

Since X ≠ 0 ⇒ |A – λI| = 0

Thus every characteristic root λ of a matrix A is a root of its characteristic equation.

33. Prove that the two matrices A and P^{-1} AP have the same characteristic roots and hence show that square matrices AB & BA have same characteristic roots if at least one of them is invertible.

Sol. $P^{-1}AP = B$

so $B - \lambda I = P^{-1}AP - \lambda I = P^{-1}AP - P^{-1}\lambda IP = P^{-1}(A - \lambda I)P$

⇒ $|B - \lambda I| = |P^{-1}|\,|A - \lambda I||P| = |A - \lambda I|$

as P^{-1} ABP and AB have same characteristic roots by assuming P = A we get

P^{-1} ABP = A^{-1} ABA = BA ⇒ BA and AB have same characteristic roots.

34. If q is a characteristic root of a non singular matrix A, then prove that $\frac{|A|}{q}$ is a characteristic root of a adj A.

Sol. AX = qX ⇒ (adj A)(AX) = (adjA)(qX) ⇒ |A| I X = q(adjA)X

⇒ |A|X = q(adjA)X ⇒ (adjA)x = $\frac{|A|}{q}$x

35. Show that if $\alpha_1, \alpha_2, ..., \alpha_n$ are n characteristic roots of a square matrix A of order n, then the roots of the matrix A^2 be $\alpha_1^2, \alpha_2^2, ...\ \alpha_n^2$.

Sol. AX = αX ⇒ A(AX) = A(αX) ⇒ $\alpha^2X = \alpha AX = \alpha.\alpha X = \alpha^2X$

⇒ $A^2 X = \alpha^2 X$ hence proved

अनुच्छेद (प्रश्न संख्या 36 से 38)

एक अशून्य सदिश X, को आव्यूह A, का अभिलाक्षणिक सदिश कहा जाता है यदि एक संख्या λ इस प्रकार विद्यमान है कि $AX = \lambda X$ तथा λ को आव्यह A का अभिलाक्षणिक मूल सदिश X के संगत कहा जाता है तथा विलोमत।

तथा $AX = \lambda X \Rightarrow (A - \lambda I)X = 0$

चूंकि $X \neq 0 \Rightarrow |A - \lambda I| = 0$

अतः आव्यूह A का प्रत्येक अभिलाक्षणिक मूल, इसकी अभिलाक्षणिक समीकरण का मूल है।

33. सिद्ध कीजिए कि दो आव्यूह A तथा $P^{-1}AP$ के समान अभिलाक्षणिक मूल है अतः दर्शाइये कि वर्ग आव्यूह AB और BA के भी समान अभिलाक्षणिक होंगें यदि उनमें से कम से कम एक प्रतिलोमिय है।

Sol. $P^{-1}AP = B$

अतः $B - \lambda I = P^{-1}AP - \lambda I = P^{-1}AP - P^{-1}\lambda IP = P^{-1}(A - \lambda I)P$

$\Rightarrow |B - \lambda I| = |P^{-1}|\,|A - \lambda I||P| = |A - \lambda I|$

चूंकि $P^{-1}ABP$ और AB के समान अभिलाक्षणिक मूल है $P = A$ पर

$P^{-1}ABP = A^{-1}ABA = BA \Rightarrow$ BA और AB के समान अभिलाक्षणिक मूल है।

34. यदि q, एक व्युत्क्रमणीय आव्यूह A का अभिलाक्षणिक मूल है तब सिद्ध कीजिए कि $\frac{|A|}{q}$, adj A का अभिलाक्षणिक मूल है।

Sol. $AX = qX \Rightarrow (\text{adj } A)(AX) = (\text{adj}A)(qX) \Rightarrow |A|\, I\, X = q(\text{adj}A)X \Rightarrow |A|X = q(\text{adj}A)X$

$\Rightarrow (\text{adj}A)x = \frac{|A|}{q}$

35. दर्शाइये कि यदि $\alpha_1, \alpha_2, ..., \alpha_n$, n क्रम की एक वर्ग आव्यूह A के n अभिलाक्षणिक मूल है तब आव्यूह A^2 के मूल $\alpha_1^2, \alpha_2^2, ... \alpha_n^2$ है।

Sol. $AX = \alpha X \Rightarrow A(AX) = A(\alpha X) \Rightarrow \alpha^2 X = \alpha AX = \alpha.\alpha X = \alpha^2 X$

$\Rightarrow A^2 X = \alpha^2 X$ hence proved अतः सिद्ध हुआ।

36. IF threre are three square matrices A, B, C of same order satisying the equation $A^3 = A^{-1}$ and let $B = -A^{3^n}$ and $C = A^{3^{(n+4)}}$ then prove that $\det(B + C) = 0$, $n \in N$

समान क्रम की यदि तीन वर्ग आव्यूह A, B, C समीकरण $A^3 = A^{-1}$ को संतुष्ठ करते है तथा $B = -A^{3^n}$ और $C = A^{3^{(n+4)}}$ तब सिद्ध कीजिए कि $\det(B + C) = 0$, $n \in N$

Sol. $C = A^{3.3^{n+3}} = (A^3)^{3^{n+3}} = (A^{-1})^{3^{n+3}} = \left(A^{3^{n+3}}\right)^{-1} = \left\{\left((A^3)^{3^{n+2}}\right)^{-1}\right\} = \left((A^{-1})^{3^{n+2}}\right)^{-1} = A^{3^{n+2}}$

$C = A^{3^n} = -B$

$B + C = 0 \Rightarrow \det(B + C) = 0$

37. If A is a non-singular matrix satisfying $AB - BA = A$, then prove that $\det.(B + I) = \det.(B - I)$

यदि A एक व्युत्क्रमणीय आव्यूह इस प्रकार है कि $AB - BA = A$ को संतुष्ठ करता है तब सिद्ध कीजिए $\det.(B + I) = \det.(B - I)$

Sol. $|A| \neq 0$

Given दिया गया है $AB - BA = A$

Hence अतः $AB = A + BA = A(I + B) \Rightarrow |AB| = |A(I + B)| \Rightarrow |A||B| = |A|\,|B + I|$

$\Rightarrow |B + I| = |B|$ ......(1)

& तथा $AB - A = BA \Rightarrow A(B - I) = BA \Rightarrow |A||B - I| = |B||A|$

$\Rightarrow |B - I| = |B|$ ......(2)

So अतः (1) & (2) $|B + I| = |B - I|$

38. If rank is a number associated with a matrix which is the highest order of non-singular sub matrix then prove that

यदि कोटि एक संख्या है जोकि आव्यूह की उच्चतम क्रम की व्युत्क्रमणीय उप आव्यूह होती है। तब सिद्ध कीजिए कि

(i) Rank of the matrix $A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & -1 & 0 \\ 2 & -7 & 4 \end{bmatrix}$ is 2

आव्यूह $A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & -1 & 0 \\ 2 & -7 & 4 \end{bmatrix}$ की कोटि 2 है

(ii) If the matrix $A = \begin{bmatrix} y+a & b & c \\ a & y+b & c \\ a & b & y+c \end{bmatrix}$ has rank 3, then $y \neq -(a+b+c)$ and $y \neq 0$

यदि आव्यूह $A = \begin{bmatrix} y+a & b & c \\ a & y+b & c \\ a & b & y+c \end{bmatrix}$ की कोटि 3 हो, तो $y \neq -(a+b+c)$ तथा $y \neq 0$

(iii) If A & B are two square matrices of order 3 such that rank of matrix AB is two, then atleast one of A & B is singular.
यदि A और B तीन क्रम के दो वर्ग आव्यूह इस प्रकार है कि आव्यूह AB की कोटि दो है, तब A और B में से कम से कम एक अव्युत्क्रमणीय है।

Sol. **(i)** $|A| = 1(-4) - 3(16 - 0) - 2(-28 + 2) = 0$

(ii) $|A| = \begin{vmatrix} y+a & b & c \\ a & y+b & c \\ a & b & y+c \end{vmatrix}$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$ (संक्रिया $C_1 \rightarrow C_1 + C_2 + C_3$ से)

$|A| = (y + a + b + c) \begin{vmatrix} 1 & b & c \\ 1 & y+b & c \\ 1 & b & y+c \end{vmatrix}$

Applying $R_1 \rightarrow R_1 - R_3$

$|A| = (y + a + b + c) \begin{vmatrix} 0 & 0 & -y \\ 1 & y+b & c \\ 1 & b & y+c \end{vmatrix} = (y + a + b + c)\, y^2 = 0$

If Rank is 3 then $y \neq 0$ and $y + (a + b + c) \neq 0$

(iii) $|AB| = 0 \Rightarrow |A| . |B| = 0 \Rightarrow$ Either $|A| = 0$ or $|B| = 0$